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PHYSICAL DEPARTMENT
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Applied Optics

VOLUME I

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Applied Optics

The Computation of Optical Systems

PHYSICAL DEPARTM AT
UNIVERSITY COLLEGE
NOTTINGHAM

Being the
"HANDBUCH DER ANGEWANDTEN
OPTIK" OF DR. ADOLPH STEINHEIL
AND DR. ERNEST VOIT

Translated & Edited by
JAMES WEIR FRENCH, B.Sc.

VOLUME I

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EDITOR'S PREFACE

As the authors have indicated in the title and foreword, this is a book of Applied Optics designed for the use of the practical optician.

To those who are not familiar with the industry, the term practical optician may imply merely the exercise of purely mechanical operations, such as the turning of tools to prescribed radii and the grinding and polishing of glass surfaces; operations involving but little scientific knowledge. Such an impression is incomplete. The work of a modern factory is much more complex.

Optical glass, the principal raw material of the optician, cannot be regarded as an isotropic substance. It has to be carefully examined, accordingly, both as regards homogeneity and strain, by means of definition-testing appliances or the interferometer and the polariscope.

Particulars of the refractive index and partial dispersions corresponding with each batch of glass are usually supplied by the maker, but for work of the best quality it is necessary at the present time to determine the constants for the individual pieces to a degree of accuracy that is only attainable by the skilful use of the best refractometers available.

If the glass is of good quality, and its constants are known to the required degree of accuracy, and if the workshop methods are good, the designer can rest assured that the optical system, when constructed, will fulfil the specified conditions without further modification. This is an essential condition of multiple manufacturing.

To obtain successful results the practical optician must combine both the science and the art of his profession. This is well illustrated by the joint authorship of this work. Dr. Adolph Steinheil, whose firm has a reputation for optical workmanship of the most excellent kind, represents principally the art, and Dr. Ernest Voit, Professor at the Technical High School of Munich, the science of the industry.

The trigonometrical system of computation, which was arranged in a practical form by Dr. L. von Seidel, and which was adopted in the factory of Steinheil, is now employed in most optical establishments. When the directness and simplicity of the method is considered, it is not surprising that a system established more than half a century ago should be in such general practical use to-day.

As Professor Seidel himself has remarked, no profound mathematical knowledge is necessary for the evolution of the few simple formulæ involved. The path of a ray of definite wave length can be traced step by step through a given optical system with an exactitude that depends only upon the accuracy of the data and the number of figures in the logarithms employed. Since, in most large factories, computing machines are used in place of tables of logarithms, great numerical accuracy is easily attainable. At each step the use of the same simple formulæ is repeated, and each ray of a particular kind is treated in the same manner. It will be found that even the formulæ for the treatment of oblique rays that do not lie in the axial plane are not much more complex than those relating to rays in the axial plane.

The accuracy and simplicity of the trigonometrical system, and the ease with which the results can be checked and supervised, are features of great importance. Under the algebraical system of computation it is hardly possible to attain the same accuracy, and in any case the high-order formulæ involved are of so complex a character as to be capable of use only by computers having considerable mathematical skill, and the number of competent computers of this type is necessarily limited.

Even when algebraical methods are employed, it will generally be found desirable to trace a number of particular rays trigonometrically through the system in order to make the final accurate adjustments of the design, and to test the results obtained.

It must not be thought that facility in the computations involved is all that is required for the design of an optical system. From the trigonometrical computation there are obtained the values of various aberrations expressed in linear or angular form. Experience alone will enable the designer to estimate what limits are permissible under the specified conditions, to decide what compromise will be best, and so to adjust the dimensions of the system as to obtain the desired final result with the least amount of calculation.

Much of the routine computation work of an optical factory consists in modifying an existing system to suit, for instance, small changes in the optical constants of the glass supplied. When there is no existing system that can serve as a basis, it is customary to determine the approximate radii of the surfaces by means of simple algebraical formulæ of the first order, such as are contained in most textbooks of optics.

Having found the approximate curvatures of the surfaces, and the aperture of the system being known, the thicknesses of the elements may be determined by geometrical construction, on the assumption that the over-all thickness should be as small as possible consistent with the requisite strength of the parts.

From the approximate form the final radii and thicknesses are then obtained by trigonometrical computation, repeated, if necessary, until the aberrations are within the required limits.

There does not appear to be any published book, other than the *Handbuch der Angewandten Optik*, of Steinheil and Voit, that presents a complete trigonometrical system of optical computation in a form suitable for practical use. Considering the special importance of the subject at the present time, it has been thought desirable to translate the work for the use of British opticians and others interested in the science.

The original book has been divided into two volumes. In the first volume there have been included those sections of the original which deal with the principles of optics and the derivation of the trigonometrical formulæ involved; with the determination of the forms of simple lenses; and with the calculation and discussion of the aberrations of lenses and their combinations.

A distinguishing and essential feature of the original work is the very complete system of symbols and sign convention employed. The portion of the original appendix which deals with this system has been included in the first chapter of the first volume and repeated in the second. English letters have been substituted for the German characters. Capital letters have been used to represent points, and small letters to represent lengths. In all other essential respects the original system has been adhered to. To illustrate the meaning of the symbols there has been added a series of ten diagrams that do not appear in the original. These diagrams include examples descriptive of the sign convention.

In the second volume there have been included those sections of the original which deal with the determination of refractive indexes and dispersions and the computation of achromatic prisms; with the computation of doublet objectives; and with the discussion of the aberrations of different combinations. There have also been included the original appendixes on "The Determination of the Refractive and Dispersive Powers of Various Media", by C. A. von Steinheil and L. von Seidel, and on "The Trigonometrical Formulæ for the Most General Case of the Refraction of Light by a System of Centred Surfaces", by Dr. L. von Seidel.

A detailed index has been provided for each of the volumes.

The illustrations throughout have been reproduced in colours instead of solely in black and white. This has necessitated the grouping of the illustrations on plates. It is hoped that the added clearness arising from the use of colours will more than compensate any disadvantage arising from the separation of the diagrams from the text. Owing to the indistinctness of yellow lines on a white ground, it has not been found practicable to make the colours in all cases correspond with the colours of particular rays of light.

It was the intention to provide a list of all alterations of the original text. This intention had to be abandoned, as the necessary corrections proved to be more numerous than was anticipated.

For the original computations, tables of logarithms were evidently used. In the work of checking, computing-machines have been employed. Although the latter results are the more accurate, the original figures have in general been retained, except where errors actually occurred. Only corrections of real importance have been given effect to.

Special care has been taken to avoid any modifications that might affect the original meaning or intention of the authors, and for this reason the earlier descriptive portions have been fully reproduced, although it is realized that greater conciseness might have been attained by the elimination of repeated statements.

The work of translation and of checking the calculations has been done under exceptional war conditions, which have made great demands upon the energies of all associated with the optical industry. For this reason it is felt that the work cannot be so free from errors and defects as might have been the case under more favourable circumstances. Criticisms and suggestions for the improvement of a possible future edition will accordingly be much appreciated.

In conclusion, I wish to express my thanks and indebtedness to Mr. J. Martin Strang, B.Sc., for his very valuable assistance in the general editorial work and in the preparation of the indexes, and my thanks also to the publishers, who have so readily agreed to the inclusion of features which I hope will be generally regarded as improvements.

JAMES WEIR FRENCH.

ANNIESLAND, GLASGOW,
January, 1918.

AUTHORS' FOREWORD

This Handbook of Applied Optics is intended to assist the practical optician. On the one hand, its purpose is to give him a clear impression of the behaviour of optical systems, and, on the other hand, to place him in the position of being able to compute optical systems with scientific exactitude.

The first of the above-mentioned desiderata, namely, the examination of the performance of optical systems, and particularly the properties of the images produced by them, is fully dealt with in many scientific contributions. In this connection we would instance the pioneer work of Möbius, Gauss, and Bessel, and the further developments of Helmholtz, Seidel, and others.

Of these theoretical discussions it is our intention to make use of only those that have a direct practical application. We shall investigate what are the conditions to be satisfied by a desired optical system, and consider which of these conditions are of most importance and which of lesser weight.

The other problem which the practical optician has to solve consists, as already indicated, in computing an optical system and so designing it that the desired result is attained by the simplest means and in the most complete manner. These questions are dealt with in Klügel's *Dioptrik*, which comprises the work done prior to the time of Fraunhofer, and which is not yet superseded by later works. Fraunhofer's statement that Klügel's *Dioptrik* is the only book of real use to the practical optician is still applicable to-day, and therefore a book that will make the more recent work available to the practician is not undesirable.

A rigid computation of optical systems only became possible when Fraunhofer, by his discovery of the fixed lines of the solar spectrum, showed how the qualities of glasses could be definitely expressed in figures. In consequence of this great discovery it became possible to calculate with rigid truth the position of an image corresponding with an axial object point. Further, as the result of the trigonometrical formulæ developed by Professor L. von Seidel, at the request of Dr. C. A. von Steinheil, the rigid computation of the image corresponding with an object point lying out of the axis can now be carried through with ease.

The determination of the images corresponding with object points lying both on the axis and out of it enables optical systems to be computed which produce in the field images of the best optical qualities as regards sharpness of definition, curvature of the surface on which the most clearly defined image lies, and finally as regards freedom from distortion.

The methods developed in this book for the computation of optical systems have by one of us been proved to be the most suitable during more than thirty years' employment in the actual production of optical instruments (in the workshops of C. A. Steinheil & Sons). Numerous computations made during this long period have taught us which rays should be chosen in order to obtain from the tracing of the smallest possible number of rays a true impression of the qualities of the images produced corresponding with two object points, one lying on the axis and the other out of it, and, further, in what sequence the conditions should be introduced in order to obtain the best possible formation of both images. To place this laboriously-won experience within the reach of others is the chief purpose of this book.

It is our intention to complete the handbook in three volumes. In this, the first volume, there will be shown a method established by actual experience that will enable the working optician having a moderate knowledge of mathematics to analyze the conditions to be fulfilled; to express correctly in figures the qualities of glasses; and to perform systematically the computation of optical systems that will produce correct images of finite dimensions. Finally, as examples, the computation of certain simple and achromatic lenses will be described.

In the second volume the results obtained in the first will be applied to the computation of optical combinations, while in the third volume the testing of the optical effects in the instruments so designed will be dealt with.

With this volume there are included four appendixes. The first contains the signs and characters used throughout the book, together with a list of the principal formulæ. The second and third are articles on "The Determination of the Refractive Indexes and Dispersions of Various Media", by C. A. von Steinheil and L. von Seidel, and on "Trigonometrical Formulæ for the General Case of the Refraction of Light by a system of Centred Spherical Surfaces", by L. von Seidel. These two articles, which are now out of print and the wording of which has not been altered by us, are inserted with the permission of Dr. Seidel. The fourth appendix is a table of "Differences of Sines and Arcs from 0° to 30° by Steps of 10 seconds". This table is principally of use for the elimination of small spherical aberrations.

THE AUTHORS.

MUNICH, *October*, 1890.

[NOTE.—Volumes II and III, referred to in the Authors' Foreword, were never published. The translation of the first volume is being issued in two portions, the second of which will comprise the chapters dealing with the computation of doublet objectives, together with the appendixes.—ED.]

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CHAPTER I

The Reflection and Refraction of Light

Laws of Reflection and Refraction

In so far as they concern the computation of optical systems, the known laws in accordance with which light is propagated through various media may be summarized as follows:[‡]—

1. Light, in its passage through a medium, suffers a loss of intensity—that is, the light is *absorbed* to an extent depending upon the constitution of the medium.

A medium that permits a large proportion of the entering light to pass through, and has therefore little absorption, is said to be *transparent*. When, on the other hand, the medium permits none of the light to pass through, it is said to be *opaque*.

The absorption of individual components of non-homogeneous light, such as white light, varies with the constitution of the medium.

If a medium absorbs equal amounts of all the components of the entering light it is said to be *colourless*. If, however, some individual components of the light are absorbed to a greater extent than others, the medium is said to be *coloured*.

2. When light is incident on the surface of separation of two media, a portion—*the reflected light*—remains in the first medium. The other portion—*the refracted light*—passes into the second medium.

3. If the surface of separation is rough, the reflection as well as the refraction of the light takes place irregularly. The light is said to be *scattered*. At a smooth surface of separation, on the other hand, the reflection is regular, as in the case of a

---

[‡] The law of the linear propagation of light in a homogeneous medium might with advantage be included. It is of interest to note that this fundamental property of light is also omitted from the list of axioms of Newton, *vide* Newton's *Opticks*, 1704.—Ed.

mirror; the refraction also is regular, and both take place in accordance with definite laws.

Since generally, in the case of optical systems, only polished surfaces of separation of two media are in question, the regular reflection and refraction of light will alone be considered in the following pages.

4. Both the reflected and the refracted rays proceed in a plane which contains the incident ray and the normal to the surface through the point of incidence, called the *normal.* If the plane containing the normal and the incident ray is called the *plane of incidence*, and the plane containing the normal and the reflected ray is called the *plane of reflection*, and if the plane containing the normal and the refracted ray is called the *plane of refraction*, then, in accordance with the above, the planes of incidence, reflection, and refraction are one and the same plane.

5. In the special case when the incident light is perpendicular to the surface of separation of the two media, then the reflected and the refracted rays coincide in direction with that of the incident ray, and they accordingly suffer no deviation.

6. The reflected ray of light remains always on the same side of the reflecting surface as the incident light, but it lies on the side of the normal remote from that of the incident ray.

The *angle of reflection*—that is, the angle between the reflected ray and the normal—is equal to the *angle of incidence*, that is, the angle between the incident ray and the normal.

The angle between the reflected and the incident ray—that is, the *angle of deviation*—is twice the angle of incidence. This is expressed algebraically as follows:—

$$(1) \qquad \phi_{2s} = \chi_{2s}.$$

$$(2) \qquad \delta_{2s} = \phi_{2s} + \chi_{2s} = 2\phi_{2s}.$$

7. Light that is not homogeneous—for example, white light—when incident upon the surface of separation of two media, is deviated by the subsequent medium, and at the same time is divided up into rays of different colours, that is to say, is *dispersed*, except when the incident light is normal to the surface, in which case the rays of incident white light are neither deviated nor *dispersed.* Certain definite regions of the colour spectrum of white light resulting from dispersion have been

denoted by Fraunhofer by letters, A, B, . . . H, which are more fully dealt with in the second volume.

8. For a particular substance the amount of refraction is different for each of the individual colours into which non-homogeneous light is divided, and in different materials the refraction varies even for the case of rays of the same colour. This property of refracting light may be expressed by a number which is known as the *refractive index* of the material in question, and which is different for different colours. Consequently, when the refractive index of a medium is stated, it is also necessary to state the colour to which the index applies.

The quotient of the refractive indexes of two consecutive media — that is, $\frac{n_{2s+1}}{n_{2s-1}}$ ‡ — is called the *refraction ratio* of the media.

The difference of the refractive indexes of a medium for two different colours, for example,

$$(3) \qquad {}_{G}n_{2s-1} - {}_{F}n_{2s-1} = \Delta_{F,G}n_{2s-1},$$

is called the *dispersion* of the medium for the two colours in question.

9. A ray after refraction proceeds on the farther side of the refracting surface in the plane of incidence extended on that side. When the subsequent medium is optically the denser, the deviation of the refracted ray takes place in a direction *towards* the normal, and when the subsequent medium is optically less dense than the preceding medium the deviation takes places *away from* the normal.

10. The angle which the refracted ray makes with the normal—that is, the *angle of refraction*—is determined from the law that the ratio of the sine of the angle of refraction to the sine of the angle of incidence is equal to the ratio of the refractive index of the first medium in which the incident ray lies to the refractive index of the subsequent medium in which the refracted ray lies. This is expressed by the formulæ:

$$(4) \qquad \frac{\sin \phi_{2s}}{\sin \phi'_{2s}} = \frac{n_{2s+1}}{n_{2s-1}}.$$

and

$$(5) \qquad \delta_{2s} = \phi_{2s} - \phi'_{2s}.$$

---

‡ The use of the suffixes employed throughout is explained in Chapter I, p. 6.—Ed.

It will be evident that reflection may be regarded as a special case of refraction. If, for instance, the refraction ratio

$$\frac{n_{2s+1}}{n_{2s-1}} = -1 \text{ and } \phi'_{2s} = \chi_{2s},$$

then the refraction formula becomes transformed into the reflection formula.

When the first medium is optically denser than the subsequent one, that is, when n_{2s-1} is greater than n_{2s+1}, or $\frac{n_{2s-1}}{n_{2s+1}}$ is greater than 1, then *total reflection* can take place; that is to say, at a certain angle of incidence the whole of the light remains in the first medium and none of it passes into the subsequent medium.

If for the angle of incidence there is chosen a value ϕ_{2s} such that

$$(6) \quad \sin \phi_{2s} = \frac{n_{2s+1}}{n_{2s-1}},$$

then

$$(7) \quad \sin \phi_{2s} \cdot \frac{n_{2s-1}}{n_{2s+1}} = 1,$$

and, from Equation (4), it follows that

$$\sin \phi'_{2s} = 1, \text{ or } \phi'_{2s} = 90°.$$

When the value of the angle of incidence is greater than that obtained from the equation

$$(6) \ \sin \phi_{2s} = \frac{n_{2s+1}}{n_{2s-1}},$$

then $\sin \phi'_{2s}$ is greater than 1—that is, an impossible value of the angle of refraction is obtained.

Suppose, for example, that the refractive index of crown glass for the particular colour denoted by the letter D is 1·536, and that for the colour H 1·547, and suppose the light passes from glass into air, then *total reflection will take place* when the angle of incidence for the coloured light corresponding with the letter D is 40° 37′ 13″, and when that for the light of colour H is 40° 16′ 18″.

In the case of refraction from flint glass into air when the refractive index for the colour D is 1·635 and that for H 1·671, total reflection will take place at angles of incidence greater than 37° 42′ 25″ and 36° 45′ 31″ respectively.

The angle of incidence—whose value can be obtained from the refractive indices by means of the formula (6)—is called the *angle of total reflection*. It indicates the upper limiting value of the angle of incidence at which refraction still takes place. For greater values of the angle of incidence the light no longer penetrates the subsequent medium.

In optical instruments use is made of both the reflection and the refraction of light. Those based upon reflection are known as *catoptric* instruments, and those based on refraction as *dioptric* instruments. In the following pages only dioptric instruments will be considered in detail. Since at the present time the latter are of much the greater importance, and since the results obtained from the consideration of dioptric instruments can when necessary be applied without difficulty to catoptric instruments, reflection will only be considered when it occurs in dioptric instruments.

Symbols and Sign Convention

General Remarks

The point of intersection of the first refracting surface with the axis is the origin from which the reckoning is commenced.

Media are denoted by odd numbers; surfaces of separation are denoted by even numbers. Thus, for example, in the case of a simple lens, the medium from which the ray of light comes is denoted by the suffix -1, the first surface of separation has the suffix 0, the substance of the lens has the suffix $+1$, the second surface of separation has the suffix $+2$, and the medium into which the ray passes has the suffix $+3$. (Fig. 1.)

Symbols for Constants

Constants are indicated by small italic type. Thus n, n', n'', . . . represent the indexes of refraction of the various glasses used. For the glass of weakest refraction the letter n is used, and for the next stronger the letter n', and so on.

The numerical suffixes at the right lower side of the letters indicate the medium to which the index refers; the alphabetical suffixes at the left lower side indicate the corresponding Fraunhofer line. For example:—

$_{D}n_1$ is the refractive index corresponding with the D line for a first crown-glass lens, and

$_{D}n_3'$ is the refractive index, also for the D line, of a succeeding flint-glass lens separated by an interval from the crown-glass lens.

l, l', l'' are abbreviations for $\frac{1}{n-1}$, $\frac{1}{n'-1}$, and $\frac{1}{n''-1}$.

The suffixes corresponding with the media and colours are similar to those used in the case of n, n', n''.

Δn, $\Delta n'$, $\Delta n''$, . . . denote dispersions.

The suffixes for the media and colours are as before; for example:—

$\Delta_{D,F}n_1$ is the value of the dispersion for a first crown-glass lens corresponding with a portion of the spectrum from D to F, and

$\Delta_{D,F}n_3'$ is the value of the dispersion for the same portion of the spectrum in the case of an adjacent flint-glass lens.

s or t denotes a chosen term of a series.

$2s$ or $2t$ represent even terms, and $2s-1$ or $2t-1$ odd terms.

m denotes the value of the quotient: $\frac{\text{size of image}}{\text{size of object}}$.

When $m > 1$ the object appears magnified.
When $m < 1$ the object appears minified.
When $m = 1$ the object appears of natural size.

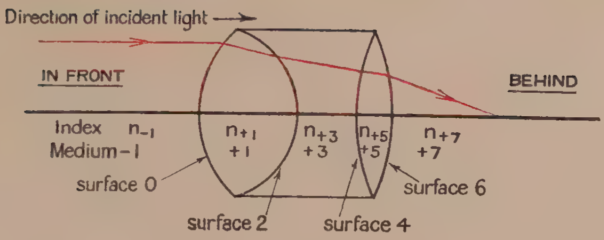


Fig. 1

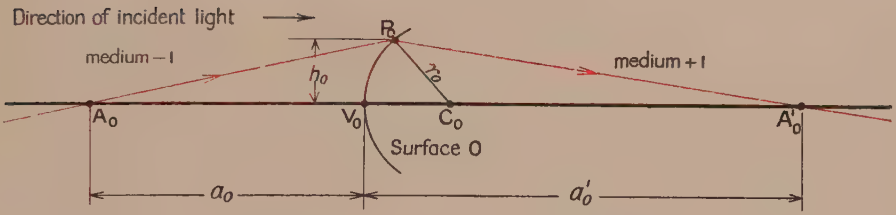


Fig. 2

a_0 negative
a'_0 positive

h_0 positive
r_0 positive

Abscissæ measured from V_0

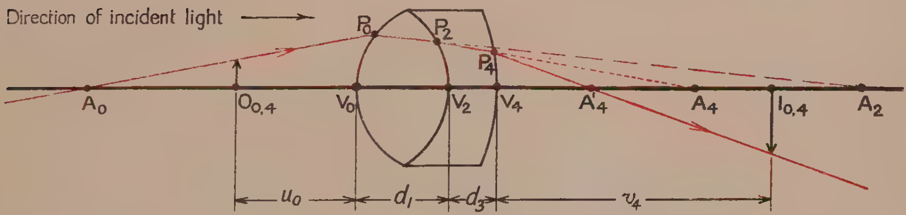
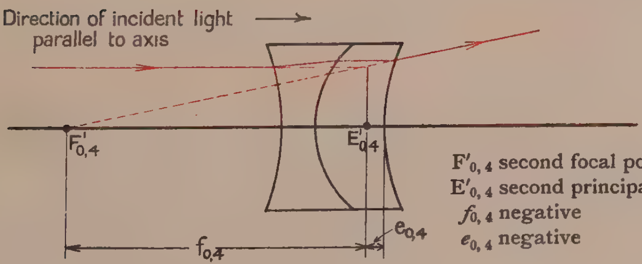


Fig. 3

a_0 negative
a_2 positive
a_4 positive

a'_4 positive
r_0 positive
r_2 negative

r_4 negative
u_0 negative
v_4 positive



$F'_{0,4}$ second focal point
$E'_{0,4}$ second principal point
$f_{0,4}$ negative
$e_{0,4}$ negative

Fig. 4

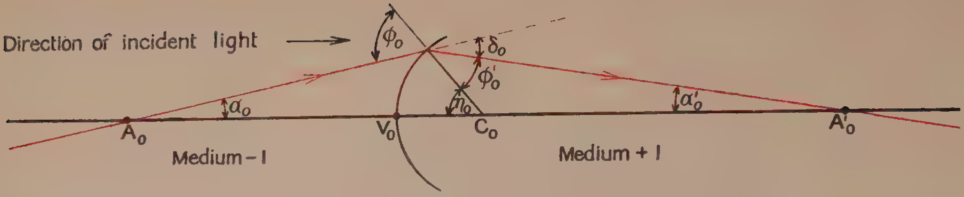


Fig. 5 α_0 negative, α'_0 positive, δ_0 positive, η_0 positive

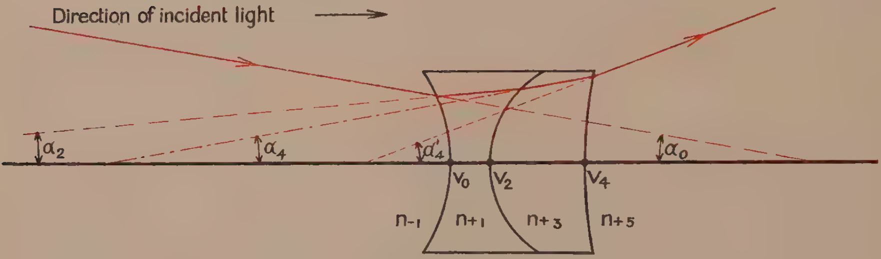


Fig. 6 α_0 positive, $\alpha_2 = \alpha'_0$ negative, $\alpha_4 = \alpha'_2$ negative, α'_4 negative

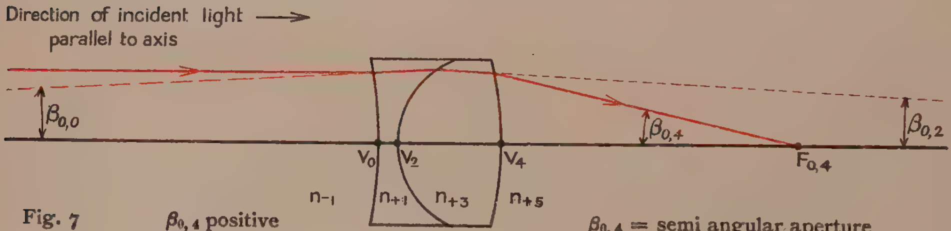


Fig. 7

$\beta_{0,4}$ positive
$\beta_{0,2}$ positive
$\beta_{0,0}$ negative

$\beta_{0,4}$ = semi angular aperture of whole system

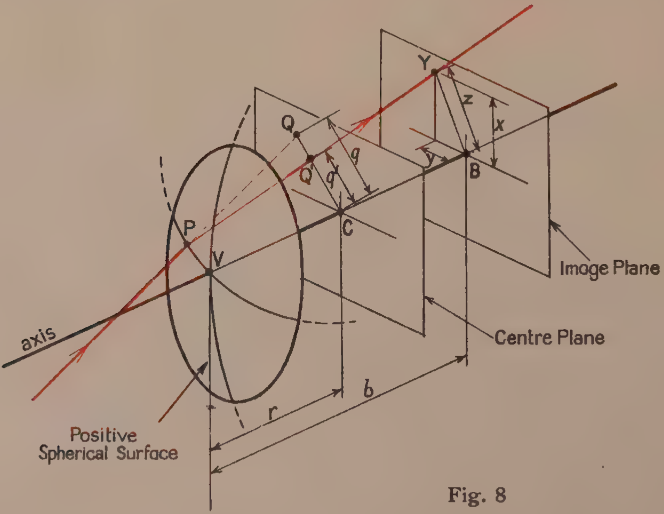


Fig. 8

Symbols for the Positions of Points

The positions of points are indicated by capital letters.

The following points, A and A′, are distinguished by the suffixes of the corresponding refracting surface:—

A denotes the point in which an *incident* ray intersects the axis (fig. 2).

A′ denotes the point in which a *refracted* ray intersects the axis (fig. 2).

The following five letters are distinguished by suffixes, separated by a comma, indicating the number of the first and last refracting surfaces of the system:—

O denotes an object point for a system of refracting surfaces (fig. 3).

I denotes the corresponding image point (fig. 3).

E and E′ denote the first and second principal points (figs. 4 and 13).

F and F′ denote the first and second focal points (figs. 4, 11, and 12).

N and N′ denote the first and second nodal points (fig. 15).

The following points are distinguished by suffixes indicating the number of the corresponding refracting surface:—

C denotes the centre of the sphere of which the refracting surface is a portion (fig. 2).

V denotes the vertex of the refracting surface, that is to say, its point of intersection with the axis (fig. 2).

P denotes the point of intersection of an incident ray with the refracting surface (fig. 2).

Q denotes the point of intersection of an *incident* ray, or its prolongation, with the plane drawn normal to the axis through the centre of the surface (*centre plane*), (fig. 8).

Q′ denotes the point of intersection of a *refracted* ray, or its prolongation, with the corresponding centre plane (fig. 8).

R denotes the point of intersection of a line drawn through the point Q, in a direction opposite to that of the incident light and parallel to the axis, with a sphere having Q as centre and any chosen radius. (See Vol. II.)

R′ denotes the point of intersection of a line drawn through the point Q′, in a direction opposite to that of the incident light and parallel to the axis, with a sphere having Q′ as centre and any chosen radius. (See Vol. II.)

S denotes the point of intersection of an *incident* ray, or its prolongation, with the above-mentioned sphere. (See definition of R.)

S′ denotes the point of intersection of a *refracted* ray, or its prolongation, with the above-mentioned sphere. (See definition of R′.)

T denotes the point in the centre plane in which the prolongation of the line CQ intersects the chosen sphere. (See definition of R.)

T′ denotes the point in the centre plane in which the prolongation of the line CQ′ intersects the chosen sphere. (See definition of R′.)

The two following letters may have suffixes 1, 2, . . . to distinguish successive image planes:—

B denotes the point of intersection of the optical axis with a chosen image plane normal to the axis (fig. 8).

Y denotes the point of intersection of a refracted ray with the image plane drawn normal to the axis (fig. 8).

Symbols for Lengths

Lengths are indicated by small italic letters.

The two following lengths are measured from the vertex of the refracting surface whose number is appended as a suffix:—

a denotes the abscissa VA of the point of intersection of an *incident* ray with the axis (fig. 2).

a' denotes the abscissa VA′ of the point of intersection of a *refracted* ray with the axis (fig. 2).

The three following letters are distinguished by the suffix of the corresponding refracting surface:—

u denotes the object distance, that is, the distance of an object point lying in the axis from the *first* refracting surface (fig. 3).

v denotes the image distance, that is, the distance from the *last* refracting surface of an image point on the axis (fig. 3).

v' denotes the distance of the best focus plane from the last refracting surface.

The two following letters are distinguished by suffixes corresponding with the medium between the vertexes of the refracting surfaces:—

c denotes generally the separation of the centre points of two consecutive refracting surfaces.

d denotes the separation of the vertexes of two consecutive refracting surfaces, that is, the thickness, which is always positive (fig. 3).

$\Delta d = 2r \sin^2\frac{\eta}{2}$. This versine of the semi-aperture angle of a surface is distinguished by a suffix corresponding with the surface in question.

The three following letters are distinguished by suffixes separated by a comma, corresponding with the first and last refracting surfaces:—

e denotes the distance of the principal point from the vertex of the last refracting surface (fig. 4).

f denotes the true focal length EF of a system of refracting surfaces (fig. 4).

g denotes the distance of the nodal point from the vertex of the last refracting surface.

The five following letters are distinguished by suffixes indicating the corresponding refracting surfaces:—

h denotes the height above the axis of the point of intersection of a ray with a refracting surface (fig. 2).

h' denotes the height above the axis of the point of intersection of a ray with the focal plane.

r is the radius of a refracting surface (fig. 2).

q is the radius vector CQ in the centre plane (fig. 8).

q' is the radius vector CQ′ in the centre plane (fig. 8).

u^* denotes the height of an object.

v^* denotes the height of an image.

The three following letters are distinguished by suffixes indicating the successive image planes:—

b denotes the distance VB of a chosen image plane, normal to the axis, from the last refracting surface.

x and y are the rectangular co-ordinates of a point Y in which an emergent ray intersects the chosen normal image plane. The value of x is measured upwards from the point B, and the value of y is measured towards the side (fig. 8).

$z = \sqrt{x^2 + y^2}$ denotes the radius vector BY in the image plane normal to the axis (fig. 8).

Symbols for Angles

Angles are indicated by small Greek letters, and are distinguished by the suffixes of the corresponding refracting surfaces.

α denotes the angle which an *incident* ray in an axial plane makes with the axis (fig. 5).

α' denotes the angle which a *refracted* ray in an axial plane makes with the axis (fig. 5).

β denotes the semi angular aperture for a ray parallel to the axis and incident upon a system of refracting surfaces. It represents the angle which the ray incident parallel to the axis makes with the axis after each refraction. This semi angular aperture has suffixes separated by a comma, indicating the first and last refracting surfaces. The angle β is a special case of α', when $\alpha = 0$ (fig. 7).

ϵ denotes the semi angular aperture of an incident ray parallel to the principal ray and lying out of the plane containing the axis. It is the angle which the ray originally incident parallel to the axis makes after each refraction with the direction of the incident principal ray. The angle ϵ for rim rays out of the axial plane corresponds with the angle β for rim rays of an incident beam that is parallel to the axis (fig. 9).

ϕ denotes the angle of incidence (fig. 5).

ϕ' denotes the angle of refraction (fig. 5).

$\delta = (\phi - \phi')$ denotes the angle of deviation (fig. 5).

γ denotes the total angle of deviation of an incident ray by a prism. (See Vol. II.)

ψ denotes the angle of a test prism. (See Vol. II.)

ϑ denotes the angle of a second test prism to be combined with the first. (See Vol. II.)

χ denotes the angle of reflection.

η denotes the angle which the radius makes with the optical axis (fig. 5).

λ denotes the angle which an *incident* ray in the plane of incidence makes with the line CQ (the internal angle at Q in the triangle PQC) (fig. 10, p. 18).

λ' denotes the angle which a *refracted* ray in the plane of incidence makes with the line CQ′ (the internal angle at Q′ in the triangle PQ′C) (fig. 10).

τ denotes the angle which an *incident* ray makes with the line drawn parallel to the axis through Q (fig. 10).

τ' denotes the angle which a *refracted* ray makes with the line drawn through Q′ parallel to the axis (fig. 10).

The following four angles are reckoned from a vertical line drawn upwards, and are measured from 0° to 360° in a clockwise direction when viewed against the direction of the incident light:—

π denotes the angle which the projection of an *incident* ray on the centre plane makes with the vertical drawn upwards at the point Q (fig. 10).

π' denotes the angle which the projection of a *refracted* ray on the centre plane makes with the vertical drawn upwards at the point Q′ (fig. 10).

ζ denotes the angle which the polar co-ordinates CQ and CQ′ in the centre plane make with the vertical drawn upwards through the centre C (fig. 10).

ξ denotes the angle which the polar co-ordinate BY in a chosen image plane makes with the vertical drawn upwards through the point B. This angle may be distinguished by suffixes 1, 2 . . ., indicating the image planes (fig. 8).

ω denotes the angle such that $\tan \omega = \frac{n_{s-1}}{n_{s+1}}$. This angle is distinguished by the suffix indicating the corresponding refracting surface.

Sign Convention

Lengths

The direction in which the incident light travels is reckoned as positive, and the opposite direction as negative (fig. 1).

Lengths measured along the axis or parallel to it are positive when measured from the vertex of the refracting surface in the direction of the incident light; they are negative when measured in the opposite direction (fig. 4).

A focal length is positive when, moving in the direction of the incident light, the principal point precedes the focal point, and negative in the reverse case (fig. 4).

Lengths perpendicular to the axis are positive when they lie above the axis, and negative when below (fig. 2).

A radius vector is always positive.

Radii of curvature may be positive or negative. The radius of a surface that presents its convex face towards the incident light is regarded as being positive, and that of a surface which is concave to the incident light as negative (fig. 2).

Thicknesses are regarded as being always positive.

Thus in fig. 2 the distance a_0 is negative, being measured from V_0 against the direction of the incident light, and a'_0 is positive, being measured in the direction of the incident light. The radius r_0 is positive, as the convex face is directed against the incident light. The incidence height h_0 is positive, since it lies above the axis.

In fig. 3, a_0 and u_0 are negative, being measured from their respective surfaces in the direction against that of the incident light. r_2 and r_4 are negative, since the concave surfaces are opposed to the direction of the incident light. The thicknesses, such as d_1 and d_3, are always numerically positive.

In fig. 4 the focal length $f_{0,4}$ is negative, since it precedes the principal point $E'_{0,4}$, and similarly $e_{0,4}$ is negative, since the principal point precedes the last vertex.

Angles

The semi angular aperture β is measured from the incident rim ray parallel to the axis to the refracted ray produced backwards. It is regarded as positive when, the eye being placed at their point of intersection, and looking against the direction of the incident light, the refracted ray appears above the incident ray, as in fig. 19, p. 28, and negative when it appears below, as in fig. 20.

All angles of deviation are measured from the incident ray, produced if necessary, to the refracted ray. The angle of deviation is reckoned as positive when the ray on refraction is bent towards the axis, and negative when bent from the axis.

The angle which a ray makes with the axis is positive when, the eye being placed at the point of intersection of the ray and the axis and looking against the direction of the incident light, the ray appears above the axis, and negative when below.

Thus in fig. 5 the angle α_0 is negative, because, when the eye is placed at A_0 and looks against the direction of the incident light, the ray appears to lie below the axis. The angle α'_0 is positive, because the eye, placed at A'_0 and looking against the incident light, sees the ray above the axis.

The signs of ϕ, ϕ', δ, γ, ψ, ϑ, χ, and η are determined by the signs of the lengths and angles from which they are derived. Thus in fig. 5 the angle η_0 is positive, since the elements which determine it—namely, the radius r_0 and the incidence height h_0—are both positive.

In fig. 6, p. 7, α_0 is positive because the ray appears above the axis. α_2, α_4, and α'_4 are all negative, because the rays when extended appear to the suitably-placed eye to lie below the axis.

In fig. 7 the semi angular apertures $\beta_{0,4}$ and $\beta_{0,2}$ are both positive, while $\beta_{0,0}$ is negative, in accordance with the rule.

In fig. 9, p. 18, the semi angular apertures ϵ_0 and ϵ_2, in the case of an oblique ray, are both positive under the conditions illustrated.

CHAPTER II

The Fundamental Properties of a Dioptric System

Definition and Properties of the Cardinal Points

As it is the intention to deal with only such dioptric systems as are of practical value, it will be assumed that the surfaces of separation of the media are spherical, since only such surfaces can be produced with the requisite accuracy, and since the use of other forms offers no appreciable advantages. It is hardly necessary to remark that plane surfaces of separation are to be regarded as spherical surfaces of infinite radius.

It will be presupposed that the centres of all the spherical surfaces of separation lie upon a straight line. With any other arrangement the requisite accuracy would be difficult of attainment and it could only give rise to serious difficulties.

A system of spherical refracting surfaces whose centres lie upon a straight line is called a *centred system.* The straight line upon which the centres lie is called the *optical axis*, and the points of intersection of the optical axis with the spherical surfaces are called the *vertexes of the surfaces*; they will be denoted by the letter V.

A dioptric system is fully determined when the refractive indexes of all the media involved and the positions of the vertexes and the centres of the surfaces of separation are known. Instead of the positions of the vertexes and the centres it is sufficient to know the position of the first vertex, the axial thicknesses of the various media, and the radii of the surfaces of separation.

An exact determination of the path traversed by a ray of light passing through a centred dioptric system can be made by the use of trigonometrical formulæ that involve no difficult mathematical operations, and the accuracy of the determination is limited only by the number of significant figures of the logarithms employed.‡ Any results obtained by computation for one particular system cannot readily be made to apply to another system of a different character. It is accordingly desirable, in the first instance, by certain limitations of the conditions, to obtain simple formulæ that will apply strictly to the limited system, which will then serve as a basis for the computation of the more complex system that is actually desired.

In the preparation of these simple approximate formulæ the following assumptions are made:—

1. Rays of light passing through the system of refracting surfaces are inclined to the optical axis only at small angles, the sines and tangents of which may be regarded as equal to the angles themselves, and the cosines of which may be regarded as unity.
2. The radii to the points of intersection of a ray of light with the refracting surfaces are inclined to the optical axis only at small angles, the sines and tangents of which may be regarded as equal to the angles themselves, and the cosines of which may be regarded as unity.

The second assumption is equivalent to the condition that the incidence height is small in comparison with the radius. By *incidence height* is meant the perpendicular distance from the optical axis of the point of intersection of the ray with the spherical surface.

3. For simplicity it will be assumed in the first instance that the entering light is homogeneous (monochromatic); that is, that it comprises only rays of one colour.

Having adopted the above-mentioned conditions, it is possible with ease to determine the path of a ray of light through a system of refracting surfaces and to represent clearly the resulting effects. The possibility of doing so is due very largely to the investigations of Gauss.‡ In the following pages there will be enumerated only those results of the Gauss theory that will be used later. For the foundation and development of the theory, the reader is referred to the under-mentioned works.§

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‡ When the amount of computation work is considerable, the use of calculating-machines is strongly recommended in place of tables of logarithms.—Ed.

It should be noted that the results given are for a system of refracting surfaces in which the first and last media are identical; that is to say, for the special case that arises in practical optics almost without exception.

For the general case, in which the first and last media are different, certain of the results must be expressed in other forms, which will be referred to as occasion arises.

Any lens system to which the above-mentioned limitations are applied has the property that all rays proceeding from one point, called the *object point*, are so deviated in their passage through the system that they meet in another single point called the *image point*. Further, if the object point lies in a plane at right angles to the axis, called the *object plane*, the image point will also lie in a plane at right angles to the axis, called the *image plane*. An object point on the axis will be denoted by the letter O, an image point on the axis by the letter I; an object point occupying a position not on the axis will be denoted by the letter O′, and an image point lying out of the axis by the letter I′.

From a consideration of the image points corresponding with certain special cases of object points, a general idea of the performance of a system of refracting surfaces can be obtained with extraordinary ease.

1. If the object point lies on the optical axis at an infinite distance in *front* of the lens system, the rays of light proceeding from the object to the system will form a beam of parallel rays

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‡ "Dioptrische Untersuchungen", von C. F. Gauss, *Abhandlungen der kgl. Gesellschaft der Wissenschaften zu Göttingen*, vol. i, 1838–43 (published 10th December, 1840); C. F. Gauss Werke, vol. v, p. 243, Göttingen, 1867; *Göttingen gelehrte Anz.*, 1841, 1.

§ An elementary treatment of the Gauss Theory is contained in *Die Haupt und Brennpunkte eines Linsensystems*, by C. Neumann. Leipzig, 1866. A very complete discussion of the subject is given in the work of Gallileo Ferraris, *Le proprietà cardinali degli strumenti dioptrici, esposizione elementare della teoria di Gauss e delle sue applicazioni*. Torrino, 1877. Authorized German edition by F. Lippich. Leipzig, 1879.

lying parallel to the axis, as in fig. 11, p. 19. The particular ray of this beam which coincides with the optical axis will not suffer refraction in its passage through the system, but all other rays will be so refracted as to intersect the optical axis in one point. This point, which will be denoted by the letter F′, is called the *second focal point* of the lens system. The second focal point is accordingly the image point of an axial object point situated at an infinite distance.

The entering rays lying parallel to the axis intersect the corresponding emergent rays of the system, when both are suitably extended, at points which lie upon a plane normal to the optical axis, and called the *second principal plane*. The point of intersection of the optical axis with this plane is called the *second principal point*, and it will be denoted by the letter E′.

2. If the beam of rays parallel to the axis and incident upon the system proceeds from an object point in the optical axis situated at an infinite distance *behind* the lens system, as in fig. 12, p. 19, i.e. in a direction opposite to that of the entering light of the first case, then the ray which coincides with the axis will be unrefracted in its passage through the system, while the other rays will intersect the axis at a point F, called the *first focal point*, situated in *front* of the system.

The incident rays parallel to the axis emerge from the system as if originally they had proceeded to a plane normal to the axis—the *first principal plane*, which intersects the axis in the *first principal point* E—and thence had been refracted towards the first focal point.

The distance from the first principal point E to the first focal point F is called the *first focal length*, and the distance from the second principal point E′ to the second focal point F′ is called the *second focal length*. Focal lengths are positive when, moving in the direction of the incident light, the principal point precedes the focal point.

In a lens system the two focal lengths have the same absolute value, but their signs are different, i.e. $f' = -f$.

3. If the object point is at an infinite distance in *front* of the system and *does not lie in the axis*, the parallel beam of rays proceeding from this point to the system will be inclined to the axis, as in fig. 13, p. 19, instead of being parallel to it as before

One of these rays, called the *principal ray*, will, after its passage through the system, be inclined to the axis at an angle equal to that at which it entered the system. The other rays will intersect the principal ray in the point where it meets the plane normal to the axis and containing the second focal point F′, that is *the second focal plane*. Further, one ray, the principal ray, of the entering oblique beam of rays, proceeds towards the first principal point E, and leaves the system as if it proceeded from the second principal point E′. The emergent principal ray is displaced relatively to the entering ray in a direction parallel to the axis by the amount of the separation of the two principal points. The other rays of the entering beam pass through the system as if originally they proceeded to the first principal plane, and then, without alteration of their relative positions in this plane, were transferred in a direction parallel to the axis into the position of the second principal plane, from whence they are so refracted as to intersect the second focal plane in the same point of intersection as the principal ray.

4. If the object point *does not lie in the axis*, and is situated at an infinite distance *behind* the system, as in fig. 14, p. 20, then one ray, the principal ray, of the parallel beam of light inclined to the axis will leave the system at an angle equal to that of the entering ray, while the other rays will be so refracted as to intersect the principal ray in a common point lying in a plane normal to the axis and passing through the first focal point F, that is in *the first focal plane.*

The principal ray of the beam approaches the second principal point E′, and thence proceeds as if it were displaced in the direction of the axis by an amount equal to the separation of the principal points, finally emerging from the system as if it proceeded from the first principal point E. The other rays pass, as if unrefracted, to the second principal plane; then, without alteration of their relative positions, they proceed as if displaced in a direction parallel to the axis, to the position of the first principal plane, and from this plane they are so deviated as to intersect the principal ray in one point lying in the first focal plane.

In the case of the beam of rays inclined to the axis, the principal points serve as direction points for the principal rays, and the object when viewed from the one principal point appears at the same angle as the image viewed from the other principal point.

From the above-mentioned considerations the general rule may be formulated that, in the case of any ray incident upon a lens system, the passage of the ray takes place as if it proceeded without deviation to the first principal plane, thence parallel to the direction of the axis, till it meets the other principal plane, from whence it is refracted so as to pass through the image point.

When the first and the last media are different, as, for example, in the case of the eye, certain of the statements in paragraphs 3 and 4 require modification. If a parallel beam of rays inclined to the axis is incident upon the system, as in fig. 15, p. 20, the principal ray proceeds towards a point called *the first nodal point*, N, and leaves the system as if it came from *the second nodal point*, N′. Similarly, in the case of a beam of rays from an object occupying a position at an infinite distance *behind* the system, and lying out of the axis, as in fig. 16, p. 20, the principal ray passes towards the second nodal point, N′, and leaves the system as if it came from the first nodal point, N.

In these two cases there are formed, instead of the principal points, two new points, called the *nodal points*, which serve as direction points for the principal rays; the object and image being viewed at the same angle from the respective nodal points.

If the focal and principal points are known, the positions of the nodal points can be determined. The distance from the first nodal point to the first focal point is equal to the second focal length, i.e. FN = E′F′; and the distance from the second nodal point to the second focal point is equal to the first focal length, i.e. F′N′ = EF.

When the focal points and the principal points are known, the lens system is fully determined, that is to say, it is possible to find the image point corresponding with any chosen object point. The four points are called the *cardinal points* of the system.

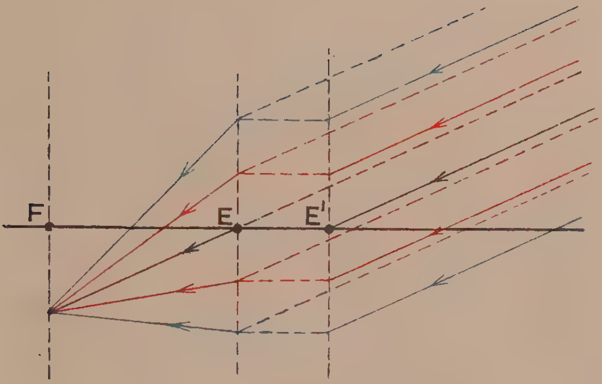

Fig. 14

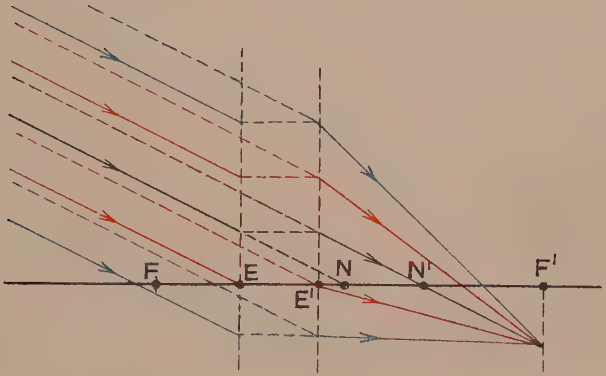

Fig. 15

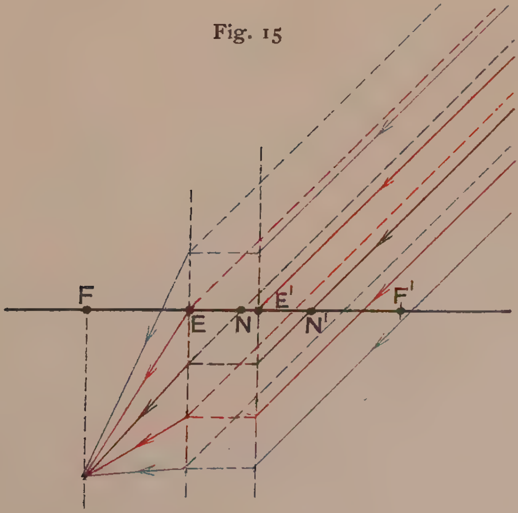

Fig. 16

[*Facing p. 20.*

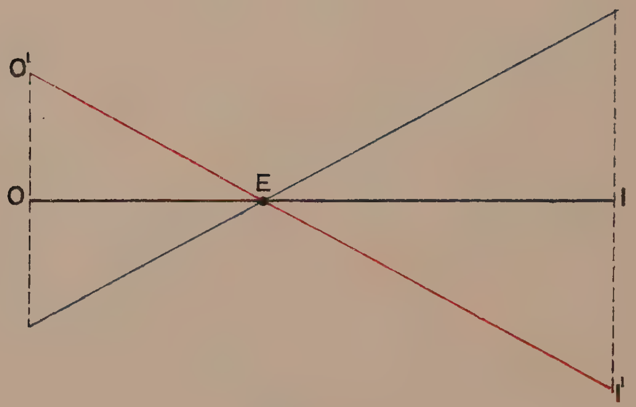

Fig. 17

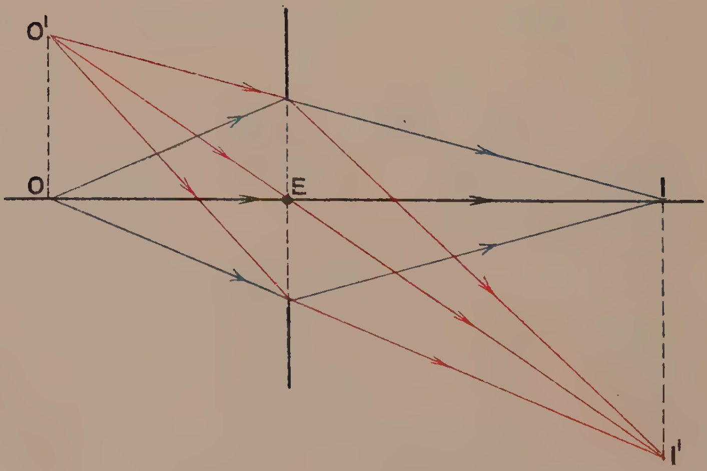

Fig. 18

Images formed by a Camera having an Infinitely Small Aperture

Before proceeding to determine from the cardinal points of various lens systems the positions of the images corresponding with chosen objects, it will be of interest to consider the case in which an image is formed without the use of any lens system. In this way some idea will be obtained of the relation of the image and object that will be of assistance when considering the images produced by lens systems.

Suppose there is set up in some position a diaphragm having an infinitely small aperture; the line normal to the diaphragm through the aperture will be called the axis. The point of intersection of the axis with the diaphragm has the property that all rays coming from an object that pass through it are principal rays, which accordingly proceed undeviated. This point is therefore the crossing point of all the rays of the system, which are all principal rays. This simple piece of apparatus may be regarded as a camera obscura which, in place of the lens, has an aperture so small that only one ray from each direction can pass through it. With such a camera the illumination is of the smallest order, but it has the remarkable property that, corresponding with any object at right angles to the axis lying in any normal plane behind the aperture, there is produced a sharply defined and true image, also lying in a plane at right angles to the axis. It is evident that all objects in front of the aperture will form images simultaneously in any normal plane behind the aperture.

The property of producing sharp images in one plane, of objects that lie in various different planes, is known as *depth of focus*. Thus the camera in question may be said to give images of infinite depth of focus. Immediately behind the aperture the size of the image is infinitely small, and the size increases in proportion to the distance of the image plane from the aperture. Since each image point is formed by only one ray, there can be no points of intersection such as may exist in the case of a pencil of rays, and therefore it is not possible to speak of focal points or focal lengths in connection with the camera in question.

The magnification, i.e. the ratio of the size of the image to that of the object, is determined, as indicated in fig. 17, p. 21, by the chosen distances of the object and image planes from the aperture. The separation of the diaphragm from the object, measured in the direction of the axis, that is, the *object distance*, will be denoted by the letter u (OE); the *image distance* will be denoted by the letter v (IE). Distances measured from the diaphragm in the direction of the incident light will be reckoned as positive; also the distances of the corresponding object and image points from the axis, i.e. the sizes of the object and image, will be denoted by the letters u^* (OO′) and v^* (II′) respectively. Distances measured upwards from the axis will be reckoned positive and those downward negative.

From fig. 17 it follows that

$$(8) \qquad \frac{v^*}{u^*} = \frac{v}{u} = m,$$

where m is the ratio of the size of the image to the size of the object. From the equation it follows that when v is greater than u the image is greater than the object, i.e. m is greater than 1; and, further, when v is equal to u, the image and object are of equal size, m being unity. The image then is of natural size. When v is smaller than u, the image is smaller than the object, and m is therefore less than 1.

Images produced by a Lens System in which the Principal Points coincide

A camera of the type described in the preceding paragraph would satisfy all practical requirements with the exception that the illumination of the image would be excessively small, owing to each image point being formed by only one ray. Since in practice brighter images are required, it is necessary to abandon the condition that the aperture shall permit only single rays to pass. But when this is the case the images lose their sharpness, since the rays from an infinitely distant object remain parallel after their passage through the aperture, and accordingly are not able to form images. By the use of a lens system it is possible, however, to refract the parallel rays and make them converge.

In the first instance, a lens system will be chosen in which the principal points coincide, as, for instance, in the case of an infinitely thin lens system or a sphere. The plane containing the *common principal point* will be called the *common principal plane.* It will be assumed that the bright image is obtained by the use of an aperture coinciding with the common principal plane, the size of the aperture being comparatively small, and yet sufficiently large to permit a beam of rays to pass. In such a lens system two of the four cardinal points—viz. the two principal points—coincide. To determine the system fully it is necessary to know also the positions of the two focal points. But since, in the particular lens system chosen, the first and last media are identical, it is sufficient to know the equal distances of the focal points from the common principal point. The lens system is completely determined when the position of the common principal point and the focal length of the system are known. The common principal point determines the position and the focal length the power of the system.

It will be evident that in the lens system described the principal rays follow the same path as in the camera previously described, since the entering rays which pass through the common principal point are unrefracted in their passage through the system, any corresponding object and image points being on one straight line through the common principal point. All other rays from one object point, after their passage through the system, intersect the principal ray of the beam in a common point. The principal ray may therefore be regarded as the central frame of the emergent rays, which combine to form the image point. Around each principal ray are grouped the other rays which proceed from the same object point, and which intersect the principal ray in the image plane, forming there the corresponding image point.

In fig. 18, p. 21, three entering and emergent rays corresponding with an object point O in the axis, and with a point O′ out of the axis, are indicated. Whereas in the type of camera already described there were neither points of intersection nor foci nor focal lengths introduced, these various features are present in the lens system under consideration, because, as has been already explained, the rays of light from one point of the object are reunited in one point after their passage through the system. Thus, if the object point lies in the axis, but at an infinite distance in front of the common principal point, the emergent rays will reunite at the focal point, the distance of which from the common principal point is the focal length. The common principal point for the given system is accordingly not only the crossing point of all the principal rays, but also the point from which the focal length is reckoned.

So far as the deviation of the incident rays is concerned, the lens system above described can act in a variety of ways, and it is necessary, therefore, to consider in greater detail the nature of the deviations produced.

If in fig. 19, p. 28, EF′ is the focal length of a lens system, then, as has been already shown, an incident ray of light that is parallel to the axis will apparently pass without alteration of direction until it reaches the common principal plane, from whence it will be so deviated as to pass through the second focal point F′. The angle β, which the ray that is parallel to the axis and is incident at the edge of the aperture makes with the corresponding emergent ray, is called the *semi angular aperture* of the system.

This semi angular aperture is measured from the incident ray to the refracted ray produced backwards. It is reckoned as positive when, the eye being placed at the point of intersection of the incident and refracted rays, and looking in the direction opposite to that of the incident light, the refracted ray appears above the incident ray, and as negative when it appears below the incident ray.

On the one hand, the semi angular aperture is equal to the angle of deviation of the incident ray that is parallel to the axis. This angle, as in the case of all angles of deviation, is measured from the incident ray to the refracted ray. It is reckoned as positive when the ray is deviated towards the axis, and negative when deviated away from the axis.

On the other hand, the semi angular aperture is also equal to the angle β between the axis and the refracted ray which passes through the focal point and which originally entered the system in a direction parallel to the axis. With regard to the sign of this angle, it should be noted that, when the eye is placed at the point of intersection of the ray with the axis, and looks in the direction opposite to that of the incident light, the angle of the ray is measured from the axis to the ray, and is reckoned as positive when the ray appears above the axis, and negative when it appears below.

According to fig. 19, the angle β satisfies the equation

$$(9) \qquad \tan \beta = \frac{h}{f},$$

where h is the distance of the incident ray of light from the axis to which it is parallel. This height is positive when measured from the axis upwards, and negative when measured downwards. In accordance with the conditions already specified, the angle β must be so small that the tangent is equal to the angle itself, in which case

$$(10) \qquad \beta = \frac{h}{f},$$

from which it follows that h must be small relatively to f.

If for various systems of lenses the diameter of the entering beam of rays is the same, and if, accordingly, the incidence height, which is small in relation to f, is constant, then the angle β has a value inversely proportional to f. The angle β measures *the illumination* of the system; this, as well as the field of view, can only be small, on account of the limitations that have been assumed. If the angle β is known, then either h or f must also be given before the power and illumination can be fixed, and the system be thus fully determined.

An incident ray that is parallel to the axis may be deviated towards the axis in the manner indicated in fig. 19. Such a system is reckoned as *positive*. The deviation of a ray parallel to the axis can, however, take place in a direction away from the axis, as indicated in fig. 20, p. 28, the system being then reckoned as *negative*. In the latter case the ray, after reaching the common principal plane, is so deviated as to appear to come from a point F′, which lies in front of that plane. Two systems of equal power, i.e. for which the numerical values of the angle β are equal, can nevertheless be different so far as the sign of β is concerned: the one system can be positive and the other negative. In the positive system an incident ray parallel to the axis at a height h is deviated through an angle $+\beta$, that is, towards the axis, whereas in the negative system a similar incident ray is deviated through an angle $-\beta$, that is, away from the axis; and, in accordance with the previously mentioned sign convention, the focal length of the positive system has the value $+f$, and that of the negative system the value $-f$.

The deviation β, imparted to an incident ray, having a given incidence height h, in its passage through the system remains constant and equal to β even when the incident ray is not parallel to the axis, as, for example, when it proceeds from any object situated on the axis at a finite distance from the system. For example, a ray proceeding from a given object point, O (fig. 21, p. 28), on the axis which intersects the common principal plane at a height h, appears to be so refracted at this plane as to intersect the axis at an image point I. The total deviation is then equal to the deviation β suffered by an incident ray that is parallel to the axis and has the same incidence height.

The angle β, corresponding with a given incidence height h, suffices to determine completely the performance of a system having a single principal point, for since β is the amount of the deviation produced by the system upon an incident ray at the given height, it determines the direction of the emergent ray corresponding with the incident ray.

It remains to be shown that, when the angular aperture and the linear aperture, or when the focal lengths of the system, are known, the image produced by the system, corresponding with any object, can be found.

The angle which the incident ray makes with the axis will be denoted by α, and the angle which the emergent ray makes with the axis by α'. With regard to the signs of these angles, it should be noted that when the eye is placed at the point of intersection of the ray with the axis, and looks in the direction opposite to that of the incident light, the angle between the axis and the ray is reckoned as positive when the ray appears above the axis and negative when it appears below.

In fig. 21, therefore, α is negative, and in accordance with the sign convention already adopted, the object distance u (EO) is negative, while α' and the image distance v (EI) are positive. From fig. 21, it follows at once that

$$(11) \qquad \beta = -\alpha + \alpha'.$$

Assuming, as before, that the angles α, α', and β are so small that the angle itself may be substituted for the tangent, then

$$(10) \quad \beta = \frac{h}{f},$$

$$(12) \quad -\alpha = \frac{h}{-u},$$

$$(13) \quad \alpha' = \frac{h}{v}.$$

If these values are substituted in the above equation, then

$$(14) \quad \frac{h}{f} = \frac{h}{v} - \frac{h}{u},$$

and, dividing both sides of the equation by h,

$$(15) \quad \frac{1}{f} = \frac{1}{v} - \frac{1}{u}.$$

From this equation any one of the three values can be determined when the other two are known; thus

$$(16) \quad f = \frac{uv}{u - v},$$

$$(17) \quad u = \frac{fv}{f - v},$$

$$(18) \quad v = \frac{fu}{f + u}.$$

But from these equations other important propositions can be deduced as follows:—

1. The image and object can be interchanged, since if the light falls upon the lens system in the opposite direction the equation becomes

$$(19) \quad \frac{1}{f} = \frac{1}{u} - \frac{1}{v},$$

the object distance u being substituted for the previous image distance v, and vice versa.

2. The object and image may be situated on opposite sides of the common principal plane. For example, the object may be in front and the image behind; or they may lie upon the same

side of the plane. In the first instance the image is *real*; that is to say, the emergent rays actually intersect in the image point. In the second instance the image is *virtual*; that is to say, the emergent rays do not intersect, but proceed as if they came from the image point. It is possible to say whether the image is real or virtual by considering whether or not the angles a and a' are of opposite or of the same sign: the former condition corresponds with the case in which the object and image are on different sides and the latter with the case in which they lie on the same side of the common principal plane.

3. The conjugate positions of the image and object are completely determined by the above-mentioned equation, and only a few special cases will be considered here.

Suppose the lens system is a positive one, and that the object lies in front of the common principal plane. Then, since f is positive and u negative,

$$(20) \qquad v = -\frac{fu}{f-u}.$$

It follows that v is positive, i.e. the image is real when u is greater than f, and, alternatively, v is negative, i.e. the image is virtual when u is smaller than f. In the special case when $v = -u$, it follows that $-u = v = 2f$. The image and object are then equally distant from the common principal plane, their respective distances being each double the focal length.

If the lens system is a negative one, and the object still lies in front of the common principal plane, then f and u are negative, and

$$(21) \qquad v = -\frac{fu}{f+u}.$$

It follows that v is always negative, i.e. the image is always virtual.

4. Since the object is viewed from the common principal point always at the same angle as the image, it follows that the magnification of the image relatively to the object is determined by the ratio of the image distance v to the object distance u. As is indicated in fig. 18, p. 21,

$$(22) \qquad \frac{-u}{u^*} = \frac{v}{-v^*}, \text{ or } \frac{-v^*}{u^*} = \frac{v}{-u},$$

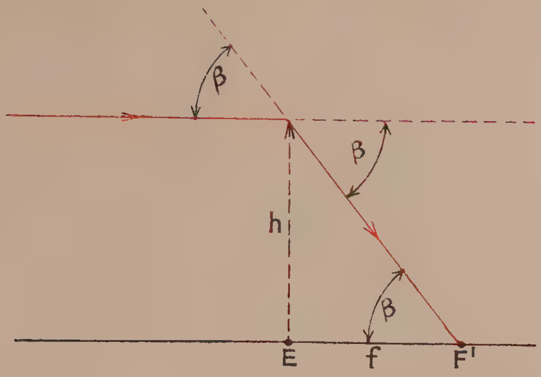

Fig. 19

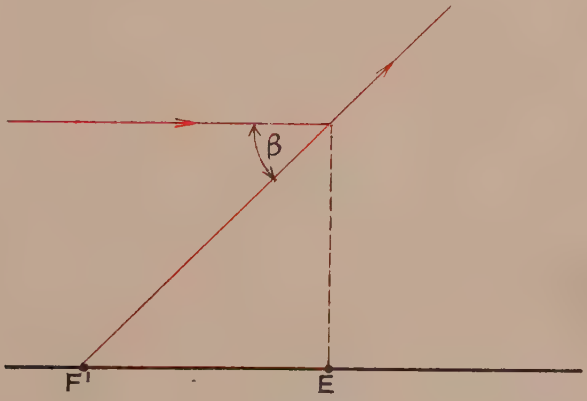

Fig. 20

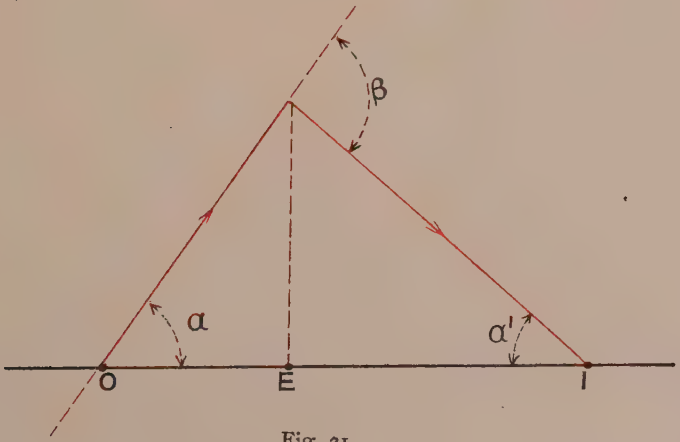

Fig. 21

[*Facing p. 28.*

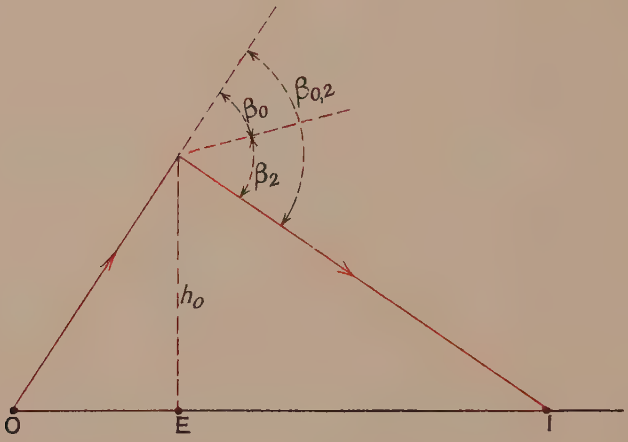

Fig. 22

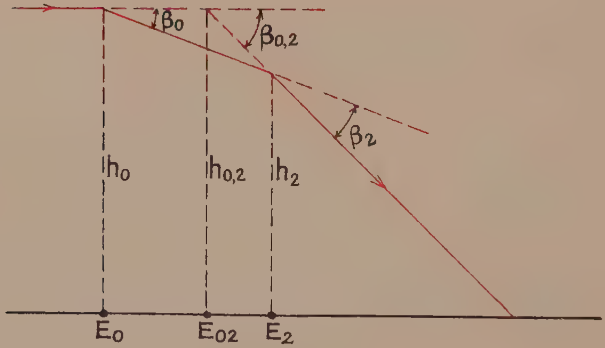

Fig. 23

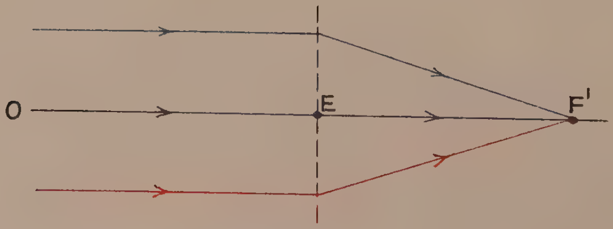

Fig. 24

and the magnification,

$$(8) \qquad m = \frac{v^*}{u^*} = \frac{v}{u}.$$

This equation can be expressed in the form,

$$(23) \qquad m = \frac{v^*}{u^*} = \frac{a}{a'},$$

since, from Equations 12 and 13,

$$(24) \qquad \frac{v}{u} = \frac{a}{a'}.$$

When, numerically, v is greater than u, it is possible to speak of the *magnification* of the image with respect to the object; when, on the other hand, v is smaller than u, the image will be smaller than the object; that is, there is *minification* of the image with respect to the object.

Finally, when v is equal to u, and therefore, as already stated, is equal to $2f$, the image and object are of equal size, i.e. the image is of *natural size.*

The above-mentioned propositions regarding the object and image may be summarized as follows:—

For a Positive System

1. When the object is at an infinite distance in front of the common principal point its real infinitely reduced image lies in the second focal point.

2. As the object approaches the system from an infinitely distant position until its distance from the common principal point is somewhat greater than twice the focal length, the real and minified image recedes from its position at the second focal point in a direction away from the common principal point to a distance somewhat less than twice the focal length.

3. When the object reaches the point the distance of which from the common principal plane is twice the focal length, its real natural-sized image lies at a distance that is also twice the focal length but behind the common principal point.

4. When the object occupies a position in front of the

common principal point at a distance which is less than twice, but greater than, the focal length, its real magnified image lies behind the principal point at a distance greater than twice the focal length.

5. When the object reaches the first focal point, its real infinitely large image lies at an infinite distance behind the common principal point.

6. When the object lies between the first focal point and the common principal point, its virtual magnified image occupies a position in front of the common principal point between infinity and the object.

7. As the object moves from the first focal point towards the common principal plane, its image moves towards the principal plane from its position at infinity in front of the system, and thus approaches the object with which it ultimately coincides when both image and object reach the common principal plane. The size of the image continuously approaches that of the object.

For a Negative System

1. When the object is at an infinite distance in front of the common principal point, its virtual infinitely-diminished image lies in front of the common principal point at a distance equal to the focal length.

2. When the object lies in front of the common principal plane, at a distance equal to the focal length, the size of the virtual image is half that of the object, and the distance of the image in front of the common principal point is equal to half the focal length.

3. As the object approaches the common principal plane its virtual image moves from the same side towards the same plane, and the size of the virtual image continuously approaches that of the object.

If, in the equation

$$(15) \qquad \frac{1}{f} = \frac{1}{v} - \frac{1}{u},$$

the focal length is made equal to 1, or, in other words, if the focal length is taken as the unit of measurement for the object and image distances, then the equation assumes the simpler form

$$(25) \qquad 1 = \frac{1}{v} - \frac{1}{u}.$$

If, in addition, the magnification, that is, the ratio $\frac{v}{u} = m$, is known, it is then always possible to determine the distances of the object and image from the common principal point; that is, to determine the dimensions u and v from the formulæ

$$(26) \qquad v = 1 - m \text{ and } u = \frac{1 - m}{m}.$$

The Appendix on page 163 indicates for various magnifying powers the object and image distances measured in terms of the focal length which is assumed to be unity. The table is provided to enable the extension of a camera corresponding with a given distance of an object from the photographic objective to be determined when the focal length of the lens combination employed and the magnification of the image with respect to the object are known. As previously stated, the object and image distances given in the tables are expressed in terms of the unit focal length. When the figures are multiplied by the focal length of the objective combination expressed in the desired unit, for example, millimetres, the object and the image distances will be found expressed in the same unit.

In the above discussions two formulæ have been used, one containing angles and the other distances. The angle formulæ were as follows:—

$$(11) \qquad \beta = -\alpha + \alpha' \quad \text{and} \quad (23) \qquad m = \frac{\alpha}{\alpha'},$$

which, it should be noted, are applicable for a given height. Those involving distances are as follows:—

$$(15) \qquad \frac{1}{f} = \frac{1}{v} - \frac{1}{u},$$

and

$$(8) \qquad m = \frac{v}{u} \quad \text{or} \quad (27) \qquad m = \frac{f}{u + f}.$$

It will be evident that in general it is more convenient in computation to use the angle formulæ than the distance formulæ, since the latter contain reciprocal values of the given and required dimensions, whereas the former contain direct values.

Summarizing the foregoing statements, it will be seen that for the determination of a lens system it is necessary to know:

1. The focal length (power).
2. The aperture (illumination).
3. The diameter of the image (field of view).

These three values together determine the performance of the lens system.

The Formation of Images by a Compound System whose Component Systems have each a Single Principal Point

From two lens systems it is possible to construct a new combined system, the power of which can be determined when those of the individual systems are known. Two cases may arise—the common principal planes of the two systems may coincide, or they may be separated by a definite distance, which will be denoted by the letter c_1, and which will be regarded as positive when traced in the direction of the incident light.

1. Suppose the common principal planes of the individual systems coincide. This is the case, for example, when both the individual systems and the combined system are infinitely thin.

Let the semi angular aperture of the first system for the incidence height h_0 (fig. 22, p. 29) be denoted by β_0; that of the second system for the incidence height $h_2 = h_0$ by β_2; and, finally, that of the combined system for the incidence height $h_{0,2} = h_0$ by $\beta_{0,2}$. The incidence heights h_0, h_2, and $h_{0,2}$ are all equal, since they lie in the same plane. It will be evident that the total deviation of the combined system is the sum of the deviations of the individual systems, i.e.—

$$(28) \qquad \beta_{0,2} = \beta_0 + \beta_2;$$

and, since

$$(10) \qquad \beta_{0,2} = \frac{h_{0,2}}{f_{0,2}}, \ \beta_0 = \frac{h_0}{f_0}, \text{ and } \beta_2 = \frac{h_2}{f_2},$$

it follows that

$$(29) \qquad \frac{1}{f_{0,2}} = \frac{1}{f_0} + \frac{1}{f_2}.$$

From the angle equation 11, and from the focal length equation 15, the power of the combined system can be determined from those of the individual systems.

It may be remarked here that the suffixes of the letters β, h, f, and c correspond with those that will be used later throughout the computations. Their meaning has already been fully explained in Chapter I.

2. Suppose the common principal planes of the individual systems are separated from one another by a distance c_1, then, as in the previous case, the total deviation of the combined system is equal to the sum of the deviations of the individual systems, but the heights at which the deviations occur in the individual systems are different, and this condition must find expression in the formulæ. As already determined,

$$(28) \qquad \beta_{0,2} = \beta_0 + \beta_2,$$

also

$$(10) \qquad \beta_{0,2} = \frac{h_{0,2}}{f_{0,2}}, \ \beta_0 = \frac{h_0}{f_0}, \ \beta_2 = \frac{h_2}{f_2},$$

and, in the case of an incident ray parallel to the axis,

$$(30) \qquad h_{0,2} = h_0;$$

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but from fig. 23, p. 29, it will be seen that

$$(31) \qquad \frac{h_0 - h_2}{c_1} = \beta_0$$

and, transposing,

$$(32) \qquad h_2 = h_0 - \beta_0 c_1.$$

Substituting this value of h_2 in equation 10, it follows that

$$(33) \qquad \beta_2 = \frac{h_0 - \beta_0 c_1}{f_2} = \frac{h_0}{f_2} - \frac{h_0 c_1}{f_0 f_2}.$$

Substituting the values obtained from equations 10, 30, and 33, in equation 28, it follows that

$$(34) \qquad \frac{h_0}{f_{0,2}} = \frac{h_0}{f_0} + \frac{h_0}{f_2} - \frac{h_0 c_1}{f_0 f_2};$$

whence, after division by h_0,

$$\frac{1}{f_{0,\,2}} = \frac{1}{f_0} + \frac{1}{f_2} - \frac{c_1}{f_0 f_2}. \tag{35}$$

From this formula is obtainable the power of the combined system in terms of the powers of the individual systems and of their separation.

The combination of three or more systems is a case that will less frequently arise. The determination of the power of a combined system, comprising any number of individual systems, offers no difficulty, however, because it is always possible to find the resultant system by combining the individual systems in pairs.

Thus, for the combination of three simple systems it can easily be shown that

1, when the principal planes are coincident

$$\frac{1}{f_{0,\,4}} = \frac{1}{f_0} + \frac{1}{f_2} + \frac{1}{f_4}, \text{ and} \tag{36}$$

2, when the common principal planes of the system are separated from one another by distances c_1 and c_2

$$\frac{1}{f_{0,\,4}} = \frac{1}{f_0} + \frac{1}{f_2} + \frac{1}{f_4} - \frac{c_1}{f_0 f_2} - \frac{c_2}{f_0 f_4} - \frac{c_2}{f_2 f_4} + \frac{c_1 c_2}{f_0 f_2 f_4}. \tag{37}$$

Constructional Method of finding the Image produced by a Lens System having a Common Principal Point

In the foregoing sections the position of an image point formed by a lens system, and corresponding with a given object point, has been determined by calculation. It is possible, also, to determine the position graphically, and the following cases will be considered.

Suppose the common principal point E and the two focal points F and F′ are given, and it is desired to determine graphically the position of the image point formed by the lens system and corresponding with an object point situated at an infinite distance, or at a finite distance, and when lying in the axis or out of the axis.

For the solution of these problems the propositions already established will be made use of. To find the image point corresponding with an object point lying *in* the axis at an *infinite* distance requires no construction, since all the rays proceeding from the point appear to proceed direct towards the common principal plane, as indicated in fig. 24, p. 29, whence they are deviated towards the second focal point. Thus the focal point F′ is the required image point corresponding with the infinitely distant axial object point.

If the object point O′, fig. 25, p. 36, lies *out of* the axis, but still at an *infinite* distance, then the rays proceeding from it pass direct to the common principal plane, where they are so deviated as to intersect the principal ray at a point which lies in the focal plane. Further, an incident ray that passes through the first focal point, and accordingly emerges in a direction parallel to the axis, intersects the focal plane in the same point. Thus the position of the image point may be found by drawing a straight line from the object point through the principal point E until it intersects the perpendicular drawn through F′. The required image point lies at the intersection of these lines. Alternatively, a straight line from the object point may be drawn through F until it meets the common principal plane, and thence parallel to the axis until it intersects the second focal plane in the required image point.

Suppose the object point O, fig. 26, p. 36, lies in the axis, but at a *finite* distance, then the rays proceeding from it pass to the common principal plane, where they are deviated towards the image point. The intersection of two of the emergent rays is the required image point. Since the axial ray is not deviated in its passage through the system, it is only necessary to determine the emergent ray corresponding with one other incident ray. For this purpose any other ray passing through the point O may be chosen, and the ray in question may be regarded as if it proceeded from an infinitely distant object lying out of the axis. It must, accordingly, on emergence from the system, pass through the point of intersection I′ of the emergent rays corresponding with the parallel rays from the object that pass through F and E respectively. Thus the position is found of the emergent ray corresponding with the chosen incident ray from the point O, and by a prolongation of the emergent ray until it cuts the axis the desired image point I is obtained.

Suppose, finally, that the object point, O′, fig. 27, p. 36, lies *out of* the axis and at a *finite* distance, then three rays proceeding from the object point can be used in the desired construction. The incident ray that is parallel to the axis is so deviated at the common principal plane as to pass through the second focal point; the incident ray that passes through the principal point proceeds in the same straight line without deviation; and finally the incident ray that passes through the first focal point is so deviated at the principal plane as to emerge in a direction parallel to the axis. These three emergent rays intersect at the image point I′. Since the intersection of two rays is sufficient to determine the image point, the third ray serves to check the result.

Lens Systems in which the Principal Points do not coincide

There will now be considered the general case of a lens system in which the two principal planes do not coincide, and in which, accordingly, there are four separate cardinal points. For the common principal plane there are now substituted two principal planes. One of these, which contains the first principal point and is normal to the axis, is the first principal plane; the other, which contains the second principal point, is the second principal plane. The separation of the principal planes has no effect on the deviation of the emergent rays relatively to the incident rays, which is the same as that produced by the system having a common principal point. That is to say, the power of the system remains entirely unaltered. But, on the other hand, the separation of the two principal planes causes the emergent rays to be displaced in the direction of the axis parallel to their original position, and the distance of the image from the object measured in the direction of the axis is increased or diminished by an amount corresponding with the separation of the two principal planes. Frequently it will be necessary in the computation of a system to determine only *the power* of the system with exactitude, the separation of the two principal planes

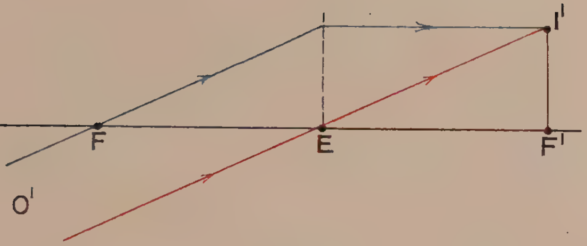

Fig. 25

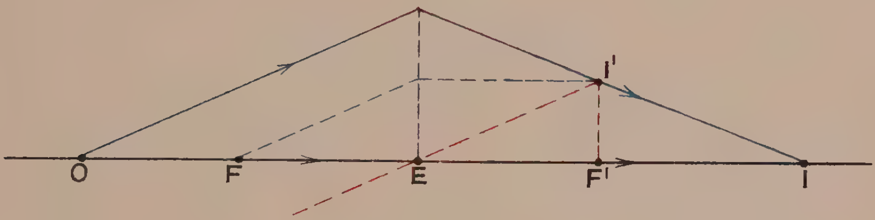

Fig. 26

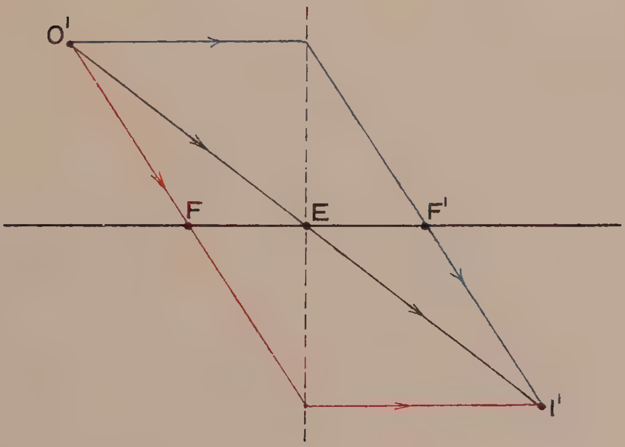

Fig. 27

[*Facing p. 36.*

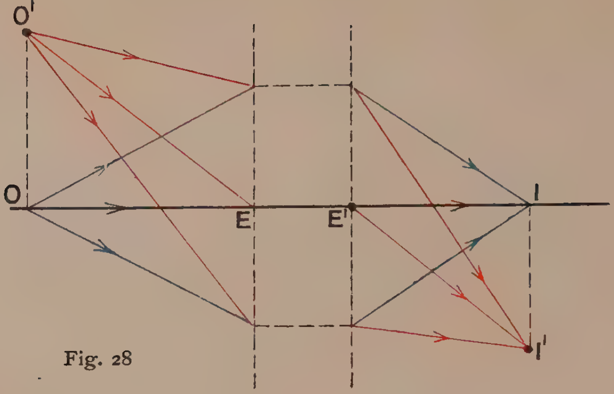

Fig. 28

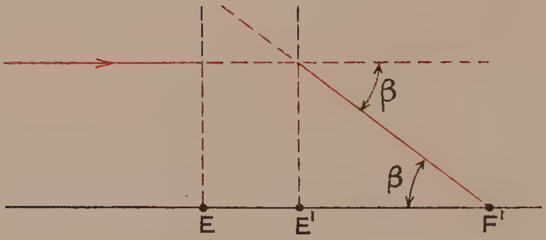

Fig. 29

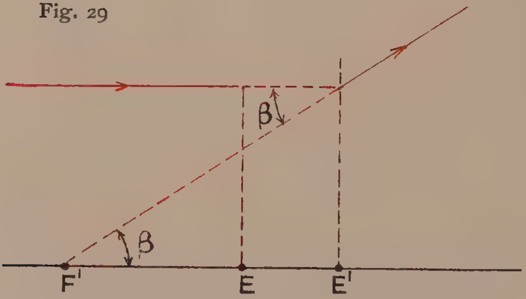

Fig. 30

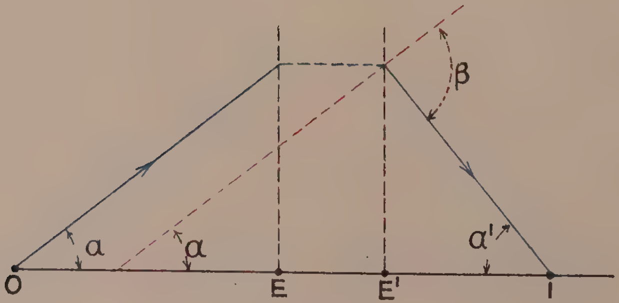

Fig. 31

being of secondary importance. This separation of the principal planes is chiefly determined by the thicknesses and also the distances between the lenses of the system. When the dimensions of these are given, the value of the separation can be found.

Images formed by a Lens System in which the Principal Points do not coincide

Some of the conclusions arrived at in the consideration of a system having a common principal point are fully applicable to a lens system in which the principal points do not coincide. The others, however, must be expressed in another form on account of the two principal planes being substituted for the common principal plane. No attempt will be made here to repeat, for the case of a system having separate principal points, all the expressions corresponding with those derived for a system having a common principal point. It will be sufficient to emphasize their points of difference and similarity.

In a system having separated principal points, the principal rays serve as a framework for the other emergent rays of the beam, but it should be noted, as indicated in fig. 28, p. 37, that all the emergent rays are displaced in the direction of the axis and parallel to their original position by an amount equal to the separation of the two principal points.

The angular aperture of the system now under consideration is defined in the manner already described. An incident ray parallel to the axis which, in the previous case of a lens system having a common principal plane, was deviated on passing through that plane, now passes direct through the first principal plane to the second principal plane, where the deviation then occurs. As will be evident from fig. 29, p. 37, the following formulæ accordingly remain unaltered, viz.:

$$(9) \qquad \tan \beta = \frac{h}{f},$$

and, when β is small,

$$(10) \qquad \beta = \frac{h}{f}.$$

As in the case of systems having common principal planes, those systems in which the principal planes do not coincide are to be regarded as positive when incident rays parallel to the axis are deviated towards the axis, as in fig. 29, and as negative when they are deviated from the axis, as in fig. 30, p. 37.

The equations already evolved for a system having a common principal point are equally applicable to a system in which the principal planes are separated. It should be noted, however, that the object distance and the first focal distance are to be measured from the first principal point, the image distance and the second focal distance being measured from the second principal point.

In the system (fig. 31, p. 37) at present under consideration the following equations are more especially applicable:—

$$(11) \qquad \beta = -\alpha + \alpha',$$

$$(23) \qquad m = \frac{\alpha}{\alpha'},$$

$$(15) \qquad \frac{1}{f} = \frac{1}{v} - \frac{1}{u},$$

$$(8) \qquad m = \frac{v}{u}.$$

The consideration of the relative position of the object and its image is equally applicable to the new system under the limiting conditions previously imposed. The power of a system resulting from the combination of two single systems can be determined from the equation

$$(35) \qquad \frac{1}{f_{0,2}} = \frac{1}{f_0} + \frac{1}{f_2} - \frac{c_1}{f_0 f_2},$$

where c_1 in this case is the distance of the second principal plane of the first single system from the first principal plane of the second single system.

Figs. 32–35, p. 40, illustrate for the type of lens system under consideration the construction for the determination of the image points corresponding with various object points. Fig. 32 shows the required construction when the object is on the axis at an infinite distance; fig. 33, when the object is at an infinite distance but out of the axis; fig. 34, when the object is on the axis and at a finite distance; and fig. 35, when the object is at a finite distance and situated out of the axis. The method of construction will be evident from the drawings, which are sufficiently self-explanatory.

When the first and last media of the system in question are not identical, there will be, as previously stated, two cardinal points in addition to the principal and focal points, namely, the two nodal points. In such a system the principal rays pass through the nodal points, and no longer through the principal points, but the latter retain in an equal degree all the other properties appertaining to them in the case of the systems previously considered, in which the first and last media are identical.

Since a system with six cardinal points is of small practical importance, no attempt will be made here to modify the various equations previously discussed, to suit the conditions arising from the presence of nodal points. It will be sufficient to indicate the construction (fig. 36, p. 41) for the determination of the image point corresponding with an object point lying in the axis at a finite distance, and the construction (fig. 37, p. 41) for the case when the object point is situated at a finite distance and lies out of the axis.

Determination of the Cardinal Points of a Given System

In the examination of optical instruments it is of interest to determine where the cardinal points lie, and accordingly the methods will be described which enable the four or six cardinal points, as the case may be, of a given system to be found by simple observations.

In the first case the positions of the two focal points and the focal distances have to be determined. From these the two principal points can be obtained. In the second case the positions of the two focal points and of the two principal points have to be determined, and from these the positions of the two nodal points can be obtained.

To Gauss is due the following observation method. From any chosen point, W, lying in the axis in front of the system, there are measured the distances to an object point lying in the axis, and to the corresponding image point. These measurements are made for three different positions of the object. Distances measured in the direction of the incident light are reckoned as positive; as, for example, the distances k_1, k_2, and k_3 from the point W to the three chosen positions of the object; the distances k'_1, k'_2, and k'_3 to the corresponding image points, and the distances l, l', from W to the first and second focal points.

In fig. 38, p. 41, the construction is indicated for one of the object points, O, and its corresponding image point, I. From a comparison of the similar triangles it follows that

$$(38) \quad -\frac{u+f}{f} = \frac{f}{v-f}, \text{ or } -(u+f)(v-f) = f^2.$$

But

$$(39) \quad -(u+f) = (l-k) \text{ and } (v-f) = (k'-l').$$

If now measurements are made of the distances of the images corresponding with different objects without making any change in the system, the three following equations may be obtained from equations 38 and 39.

$$(40) \quad (l-k_1)(k'_1-l') = f^2.$$

$$(41) \quad (l-k_2)(k'_2-l') = f^2.$$

$$(42) \quad (l-k_3)(k'_3-l') = f^2.$$

NOTE.—Remembering that f is positive for a positive system, and that u is in this case negative, since it is measured from the principal point E in the direction opposite to that of the incident light.—ED.

Since the values of k_1, k_2, k_3, k'_1, k'_2, and k'_3 are known as the result of measurement, the three unknowns, l, l', and f may be determined by calculation from the equations 40, 41, and 42. Thus

$$(43) \quad l = k_1 - \frac{(k_2-k_1)(k_3-k_1)(k'_2-k'_3)}{k_0},$$

$$(44) \quad l' = k'_1 - \frac{(k'_2-k'_1)(k'_3-k'_1)(k_3-k_2)}{k_0},$$

$$(45) \quad f^2 = \frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)(k'_2-k'_1)(k'_1-k'_3)(k'_2-k'_3)}{k_0^2},$$

where

$$(46) \quad k_0 = (k_3-k_2)(k'_1-k'_2) - (k_2-k_1)(k'_2-k'_3).$$

It is only possible to make these observations when the lens system forms a sharp image on the axis, since only then can the distance from W to the image point be measured exactly.

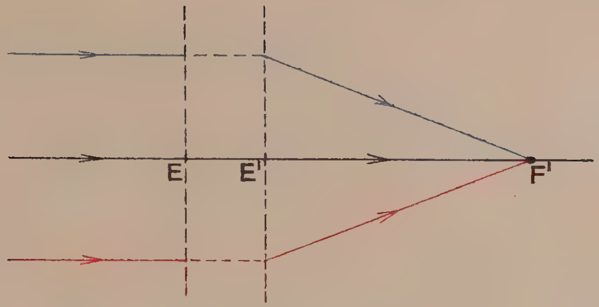

Fig. 32

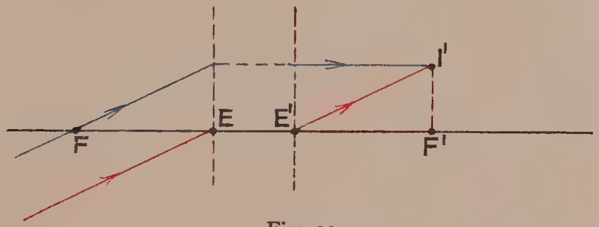

Fig. 33

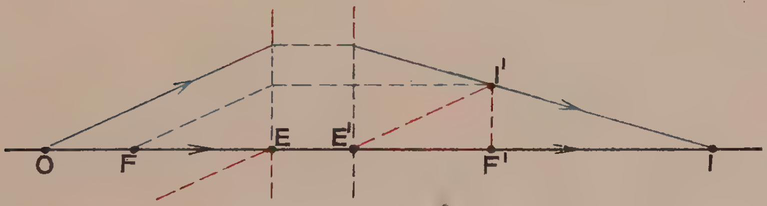

Fig. 34

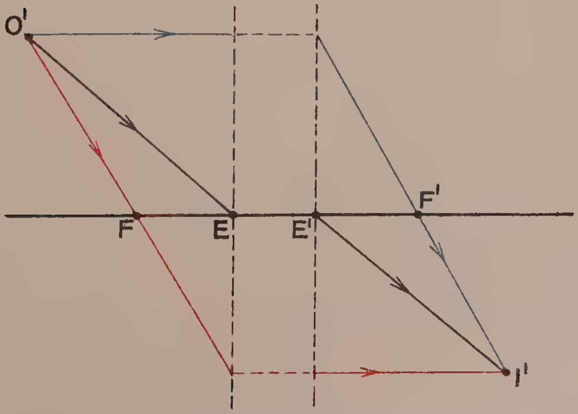

Fig. 35

[*Facing p. 40.*

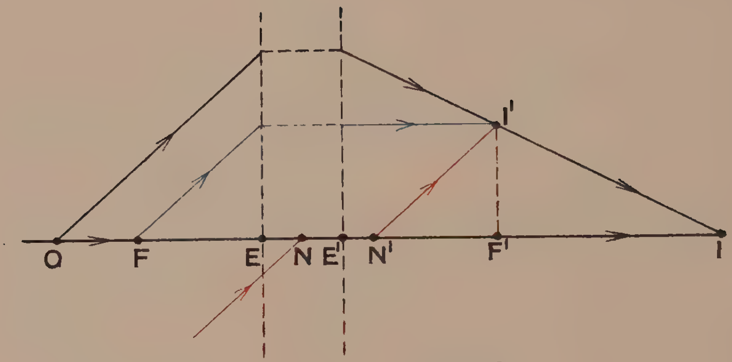

Fig. 36

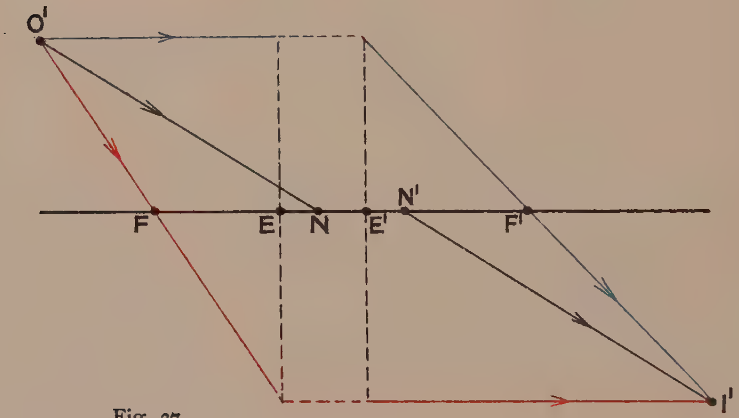

Fig. 37

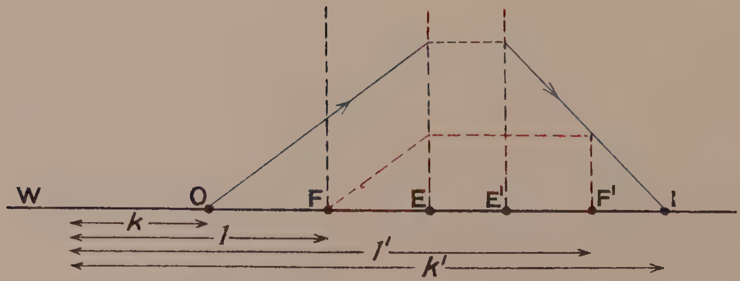

Fig. 38

Further, since a positive system alone forms a real image, the above-mentioned method can be directly applied only to such a system. It can be shown, however, that it is possible to determine the positions of the cardinal points even in the case of a negative system. To the negative system the particulars of which are required there may be added a positive system, such that the combined system will form a real image. If the cardinal points of the combined system and of the auxiliary positive system are determined in the manner above described, it is then possible to determine the positions of the cardinal points of the negative system.

If in the system, the cardinal points of which have to be found, the first and last media are different, there will be, as previously indicated, six cardinal points. These can be most readily determined by making four observations as proposed by Helmholtz.‡ From a chosen point on the axis the distances are measured to two positions of an object and to its corresponding image points, and also the sizes of the object and of the corresponding images are observed. If the distances from the point W to the first and second principal points are denoted by the letters j and j' respectively, it follows from the equations

$$(47) \qquad -\frac{f}{u}+\frac{f'}{v} = 1 \text{ and } \frac{f}{u}\cdot u^* = \frac{f'}{v}\cdot v^*,$$

that

$$(48) \qquad f = j - l \text{ and } f' = l' - j',$$

and, further, that

$$(49) \qquad -u = j - k, \text{ and } v = k' - j'.$$

The four equations obtainable from the observations are therefore as follows:—

$$(50) \qquad \frac{j-l}{j-k_1} + \frac{l'-j'}{k'_1-j'} = 1.$$

$$(51) \qquad \frac{j-l}{j-k_1}\cdot u^*_1 = \frac{j'-l'}{k'_1-j'}\cdot v^*_1.$$

$$(52) \qquad \frac{j-l}{j-k_2} + \frac{l'-j'}{k'_2-j'} = 1.$$

$$(53) \qquad \frac{j-l}{j-k_2}\cdot u^*_2 = \frac{j'-l'}{k'_2-j'}\cdot v^*_2.$$

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‡ Helmholtz, *Physiol. Optik*, par. 10, p. 78 onwards.

From these four equations the value of l, l', j, and j' can be obtained thus:—

$$(54) \qquad l = \frac{k_2 u^*_1 v^*_2 - k_1 u^*_2 v^*_1}{u^*_1 v^*_2 - u^*_2 v^*_1};$$

$$(55) \qquad l' = \frac{k'_1 u^*_1 v^*_2 - k'_2 u^*_2 v^*_1}{u^*_1 v^*_2 - u^*_2 v^*_1};$$

$$(56) \qquad j = \frac{k_1 v^*_1 (v^*_2 - u^*_2) - k_2 v^*_2 (v^*_1 - u^*_1)}{u^*_1 v^*_2 - u^*_2 v^*_1};$$

$$(57) \qquad j' = \frac{k'_2 u^*_2 (u^*_1 - v^*_1) - k'_1 u^*_1 (u^*_2 - v^*_2)}{u^*_1 v^*_2 - u^*_2 v^*_1}.$$

It will be understood that the observations of the distance and size of the image can only be made with accuracy when the system produces, not only a sharp image on the axis, but also correct images out of the axis, since only in such cases can accurate setting be obtained.

Naturally this method is applicable also to a system in which the first and last media are identical. As a check on the results the following formulæ may then be employed:—

$$(58) \quad j - l = j' - l' = \frac{k_1 v^*_1 v^*_2 - k_2 v^*_1 v^*_2}{u^*_1 v^*_2 - u^*_2 v^*_1} = \frac{k'_2 u^*_1 u^*_2 - k'_1 u^*_1 u^*_2}{u^*_1 v^*_2 - u^*_2 v^*_1};$$

or, since (48) $f = (j - l) = (j' - l')$,

$$(59) \quad f = \frac{k_1 - k_2}{\frac{u^*_1}{v^*_1} - \frac{u^*_2}{v^*_2}} = \frac{k'_1 - k'_2}{\frac{v^*_1}{u^*_1} - \frac{v^*_2}{u^*_2}}.$$

CHAPTER III

The Requirements of an Actual Lens System and the Calculation of the Errors to be eliminated

Requirements to be fulfilled by an Actual System

In the foregoing chapter three limitations were imposed:

1. That the incident light should be of only one colour, i.e. monochromatic.
2. That the rays passing through the system should be inclined to the optical axis only at small angles, for the sine and the tangent of which the arc may be substituted and unity for the cosine.
3. That the radii to the points of incidence should be inclined to the axis only at small angles, for the sines and the tangents of which the arc may be substituted and unity for the cosines.

The two last limitations are equivalent to the assumption that the diameter of the cylindrical incident beam parallel to the axis is small, and that the principal ray that passes through the system from an object point situated out of the axis is inclined to the axis at a small angle, or, in other words, that the system possesses very small illumination and has a very small angular field.

In the case of an actual lens system these three limitations can no longer be imposed, since, in general, the light that enters the system will be white and not monochromatic. Further, in order to obtain bright images the diameter of the cylindrical beam proceeding from an infinitely distant object situated on the axis must be large in proportion to the focal length; that is to say, the lens system must have a large angular aperture; and, finally, in order that a substantial portion of the object may be viewed, the system must have a large angular field.

It should also be remarked that the diameter of the cylindrical beam, proceeding to the system from an infinitely distant object situated in the axis, will determine the effective aperture of the system and also the illumination of the image point on the axis. In order that the illumination of an image point corresponding with an infinitely distant object point situated out of the axis may be equally bright, the diameter of the oblique cylindrical beam of incident rays must be equal to that of the beam parallel to the axis.

Although in the case of an actual lens system the limitations previously mentioned are no longer permissible, it is necessary that the system should produce an image corresponding as closely as possible with that of the ideal system. It should have the same magnitude, and, notwithstanding the greater illumination and angular field, the image should be as truly reproduced as that of the ideal system. The image is said to be *true*

(1) When it is sharply defined; that is, when all the rays (even of different colours) proceeding from an object point meet in one point, namely the corresponding image point;

(2) When it is undistorted; that is, when the ratio of the distance between two image points to the distance between the two corresponding object points is the same over the whole extent of the field. This condition must be satisfied for rays of different colours.

As previously stated, the magnitude, illumination, and extent of the field produced by a lens system may be regarded as the three quantities that define its performance. To determine the performance of the system it is necessary to know

(1) The focal length f;

(2) The illumination obtained from the quotient $\frac{h}{f}$, that is, the semi-diameter of the incident beam of rays parallel to the axis divided by the focal length; and

(3) The angular field of view, τ.

In the case of the actual system the *true focal length* can only be made to correspond completely with the focal length of the ideal system when the true focal length of the actual system is regarded as being the distance of the principal point from the focal point for an incident *paraxial ray* of medium refrangibility. On this assumption the axial images of the two systems would be of equal size. In addition, it is desirable that, notwithstanding the greater illumination and field of view, that is, the greater values of $\frac{h}{f}$ and τ, the images should still be correctly delineated.

If now the actual system has a large aperture relatively to its focal length, and if, notwithstanding this, well-defined images are desired, it is necessary on the one hand that incident rays lying at a considerable distance from the axis should meet in the focal point, and since, further, it is desired that the system should have only one focal length, f, it is necessary that the points at which the outer incident rays are virtually deviated on emergence should be at the same distance from the focal point as those of the axial rays; that is to say, the *virtual emergence points*‡, or the points at which the rays may be regarded as being deviated towards the focus, must lie upon a sphere drawn through the axial principal point, with the focus as centre and the focal length as radius.

It will be seen from the above that the semi angular aperture is obtainable from the equation

$$(60) \qquad \sin\beta = \frac{h}{f}.$$

If the focal length is very great, or the aperture relatively small, then the spherical surface will become more nearly a plane surface normal to the axis, in which case

$$(61) \qquad \tan\beta = \frac{h}{f}.$$

If the angular field of the system is very great it is desirable that the principal rays should cross one another at the second principal point, and that the peripheral rays of each beam should intersect one another on the principal ray at a distance from the principal point corresponding with the actual focal length of the system. The image thus produced will lie upon a spherical surface having the second principal point as centre and the focal length as radius. The semi angular field is obtained from the equation

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‡ The authors use the expression *die Anfangspunkte der Brennweiten*, that is, the commencing points of the focal lengths.—ED.

$$(62) \qquad \sin \tau = \frac{h'}{f}.[‡]$$

As the true focal length increases, and the angle τ diminishes, the spherical image surface approaches a plane normal to the axis, and when in the limit it coincides with this plane,

$$(63) \qquad \tan \tau = \frac{h'}{f}.$$

In order to find whether or not an actual lens system satisfies the conditions already referred to, it is not necessary to examine the positions of all the rays emerging from the system. By making a suitable selection only a very small number of rays need be considered.

Suppose, for example, it is desired to find the images corresponding with two object points situated at an infinite distance in front of the lens system, the first object point being on the optical axis and the second situated out of the axis. In the first case it is sufficient to choose two incident rays of white light and in the second case four rays of white light. If these rays satisfy the stipulated requirements it will be the case also, at least very nearly, for all other rays.

I. For an image point on the axis it is necessary to trace two rays of white light parallel to the axis and incident on the first lens surface, namely:

(*a*) *A paraxial ray*, that is, one lying close to the axis and parallel to it; and

(*b*) *A rim ray*, that is, one incident close to the edge of the lens.

As the result of the first refraction by the lens system each white ray will be dispersed, and thereafter two colours; that is, a total of four rays, must be traced through the system. Any two colours may be chosen provided that throughout the computation of any particular system the same two colours are adhered to.

[‡] h' being the incidence height on the spherical focal surface.—Ed.

In reckoning the individual rays trigonometrically through the lens system, the incident height h is chosen, and it is required to find: the angle β which the emergent ray makes with the axis, the distance v from the vertex of the last surface of the system to the point of intersection of the ray with the axis, that is, the *back focal length*, and also the distance from the vertex to the principal point E′.

(*a*) For the axial ray it is desirable that the two coloured rays into which it is divided by refraction at the first lens surface should intersect the axis at equal distances from the vertex of the last refracting surface. This point of intersection is the focus previously referred to. Further, it is desirable that the principal points should be the same for the two colours. If the focal points coincide, the system is free from *chromatic aberration*; if, at the same time, the principal points are coincident, then the *chromatic difference of magnification* is eliminated. The latter statement will be understood when it is remembered that, as explained in the previous chapter, the size of image is dependent upon the focal length, that is, upon the positions of the principal points when the foci coincide. When the focal lengths are equal under these conditions the images of different colours will occupy the same positions and be equally large.

(*b*) For the peripheral ray it is desirable that the two coloured rays, which should be the same as those considered in the case of the axial rays, should intersect the axis in the same point as the axial rays, that is, in the focus. It is desirable, further, that both rays should have the same true focal length as the axial rays, that is, that the virtual emergence points from which the rays bend towards the focus should lie upon a sphere described through the second axial principal point, with the focus as centre and the true focal length as radius.

If the axial and peripheral rays of the same colour intersect the axis in a common point, there is then no *spherical aberration* for rays of this colour; and if for the second colour also the peripheral ray meets the axis in the same point of intersection as the axial ray, there is then no spherical aberration for the second colour. This latter condition is called the *Gauss condition.*‡

If in the case of the peripheral ray of the first colour, not only is its focus coincident with that of the axial ray, but also the distance of the principal point (situated upon the spherical principal surface) from the focus is the same, then there is no *distortion* for this particular colour, since a system that has for both axial and peripheral rays the same true focal length must have for both axis and periphery the same magnification.

If the principal points are also the same for this second colour, the image formed by rays of the second colour will also be undistorted, and will have the same magnification as the image formed by rays of the other colour for both the periphery and the centre of the system. The image will in such a case be completely achromatic; that is, the rays for any two colours that before the first refraction were constituents of a white ray will be so refracted through the system that on emergence they will again form a white ray, and that they will emerge from the same part of the last surface of the system, and be directed towards the same point.§

II. In the case of the second image point lying out of the axis it is necessary to trace the path of four rays. There should first be selected the ray that passes undeviated through the system, that is, the principal ray. As already stated, this ray makes the same angle, τ, with the axis on incidence as it does on emergence. This particular ray, which may be regarded as the axis of the beam proceeding from the object point situated out of the axis of the system, intersects the image surface at a height, h', equal to $f \sin \tau$. Around this principal ray the entering beam is symmetrically arranged, and of this beam three peripheral rays should be selected. Suppose, for example, an image point is selected lying in a vertical plane under the axis, then there would be chosen the two peripheral rays of the beam that lie in this vertical plane, one called the *upper ray*, situated above the principal ray, and the other called the *lower ray*, situated below, and also a third peripheral ray parallel to these two, but situated at 90° from either. The last-mentioned *side*

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‡ Gauss, "Dioptrische Untersuchungen".

§ *Traité Elementaire d'Astronomie Physique*, par J. B. Biot. Paris, 1844. Volume II, p. 82.

ray of the beam is symmetrically situated as regards the two sides of the vertical plane, and the plane in which the side rays are refracted changes from surface to surface. While the upper and lower rays lie in a plane containing the optical axis, the side ray does not lie in an axial plane; that is to say, no plane can be found which simultaneously contains the optical axis and the side ray.

In a correctly designed system the image of an object point situated out of the axis must be well defined and undistorted. If the three chosen rays—namely, the upper, lower, and the side ray—intersect the principal ray in the same point, the image will be clear. If the distance of this point of intersection from the principal point is equal to the true focal length, in which case the point of intersection lies on the image surface, and if the principal ray emerges as if it came from the second principal point, then the image will be also *undistorted*; that is to say, it will appear magnified by the same amount as the axial image.

Similar requirements have to be satisfied by rays of a second colour forming part of the original incident beam of white light. If for the two colours in question this condition is satisfied simultaneously with those already mentioned, then the well-defined and undistorted spherical image will also be completely achromatic over its whole extent.[‡]

Defects of the Image produced by an Actual Lens System

In order to determine the defects of the image produced by an actual lens system it is necessary to compare it with the image produced by the equivalent ideal system, that is, the ideal system of equal focal length. The value of the equivalent ideal system may be obtained by tracing trigonometrically the passage through the system of an incident paraxial ray of medium refrangibility, and by reckoning the true focal length as being the distance between the principal and focal points. In this way the second principal point and the second focal point may be found.

[‡] In the case of systems of large angular aperture there should be selected for the determination of the image point in the axis a third ray having an incidence height of two-thirds the semi-aperture, and for the image point situated out of the axis there should be selected three additional rays on the periphery of a cylindrical beam having a diameter two-thirds the working diameter of the aperture, the upper and lower rays lying in the plane which contains the axis and the chosen extreme peripheral ray. In this way some idea as to the presence and amount of small secondary errors may be obtained.

If, similarly, there is considered a paraxial ray of medium refrangibility that enters the system from the opposite direction, the true focal length will, as before, be the distance from the principal point to the focal point. Thus the first principal point and the first focal point may be determined.

In a lens system the distance of the first focal point from the first principal point is equal to the distance from the second focal point to the second principal point, and equal to the true focal length.

When the four cardinal points, namely, the two principal points and the two foci, are known, the image corresponding with that formed by the ideal system may be found by construction. Those rays of the incident beam parallel to the axis, that are chosen for computation, must so emerge as if they had passed unrefracted to the second principal surface, and had thence been deviated towards the second focal point. Of those rays of the incident oblique beam that are chosen for computation the principal ray must so proceed as if it had passed unrefracted to the first principal point, and had then been displaced along the axis to the position of the second principal point, from whence it proceeds, in a direction inclined to the axis at the same angle as that of the incident ray, until it meets the image surface. The remaining rays of the oblique beam proceed parallel to the principal ray until they reach the second principal surface, from whence they so pass as to meet the image surface at the same point of intersection as the principal ray.

All differences between the positions of the rays actually computed and those of the above-mentioned ideal rays are called the errors or aberrations of the system. A distinction must be drawn between errors of the image points on the axis and errors of the image points lying without the axis.

In determining the errors it will be assumed that the lens is positive, and the necessary modifications to suit the case of a negative lens will only be mentioned as occasion requires. Further, in most practical cases it is sufficient to consider only the images of infinitely distant object points, which accordingly lie in the image surface containing the second focal point. Of these the first to be considered is the image corresponding with the infinitely distant object point lying in the axis. It should be reckoned for two colours, the choice of these colours depending upon the conditions to be fulfilled by the system.

In the case of those systems to be first considered, that serve as a direct aid to the eye (optical systems), a ray of light having a refrangibility corresponding with the D line will be selected for the determination of the focal and principal points of the ideal system, and for the second colour a ray of stronger refraction, for example, that corresponding with the F line, will be chosen. For photographic apparatus in which focussing of the optical image is desired, the G and D rays should be chosen. When it is desired to obtain the sharpest possible images of chemically active rays without regard to the position of the optical image, the G and H rays should be chosen. The determination of the errors of the first-mentioned optical systems will alone be considered. There need then be no difficulty in dealing with the other two cases by analogy.

Of the rays that proceed from an infinitely distant axial object it is necessary, as previously stated, to trace through the system two rays, one incident near the axis and the other incident at the periphery of the first surface. Both rays must be computed for two colours. By tracing an axial D ray the second principal point and the second focal point are determined. The axial F ray gives the error in the positions of the second principal point and the second focal point for this colour. If the more strongly refracted F ray does not intersect the axis at the same point as the D ray, there is said to be *chromatic aberration* of the axial rays. If the more strongly refracted ray cuts the axis too soon, as compared with the D ray, when reckoned in the direction of the incident light, whereby its focal length is shorter, the chromatic aberration is said to be uncorrected (fig. 39, p. 52). On the other hand, when the ray cuts the axis later, the error is said to be over-corrected (fig. 40, p. 52).

An aberration is said to be *corrected* when it is reduced to zero, *uncorrected*‡ when its sense is the same as that of a simple positive lens with equally refracting surfaces, and *over-corrected* when the value lies on the other side of zero. Uncorrected aberrations will be denoted by the sign +, and over-corrected aberrations by the sign −.

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‡ The term under-corrected is more generally used in this country.—Ed.

The distance between the principal point of the axial rays and the point of intersection with the axis of the emergent rays may be determined from the formula

$$(64) \qquad E'I = \frac{h}{\tan\beta} = \frac{h}{\beta},$$

and the distance of the principal point from the vertex of the last refracting surface is obtained from the equation

$$(65) \qquad E'I - V_2I = E'V_2 = e'.$$

If now the principal point for the F ray, measured in the direction of the incident light, lies behind the principal point for the D ray, whereby the focal length is shorter, the *chromatic difference of magnification* is said to be *uncorrected* (fig. 41, p. 52), as compared with the opposite case of *over-correction* (fig. 42, p. 52).

By tracing in this way the axial rays for two colours, there have been obtained

1. The power of the system.
2. The chromatic aberration for the axial ray.
3. The chromatic difference of magnification of the system.

If for the yellow ray the value

$$V_2F' = v,$$

and

$$E'V_2 = e',$$

then the focal length f of the system may be obtained from the formula

$$(66) \qquad f = E'V_2 + V_2F' = e' + v.$$

Further, the chromatic aberration of the axial rays will be equal to $v_{(\text{D ray})} - v_{(\text{F ray})}$ (positive when uncorrected), and the chromatic difference of magnification will be equal to $e'_{(\text{D ray})} - e'_{(\text{F ray})}$ (positive when uncorrected).

The incidence height h for the rim rays, which determines the illumination and also the angular aperture of the system, should now be chosen. The semi angular aperture is obtained from the equation

$$(60) \qquad \sin\beta = \frac{h}{f},$$

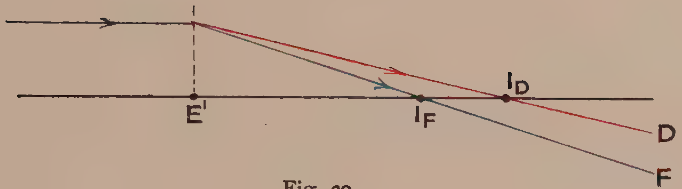

Fig. 39

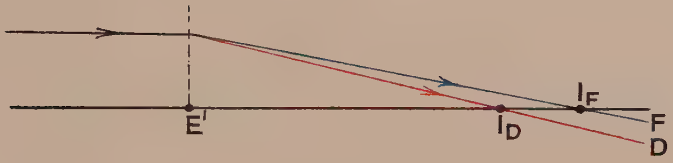

Fig. 40

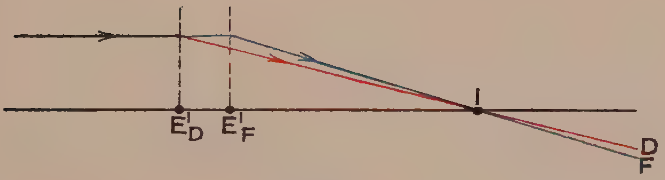

Fig. 41

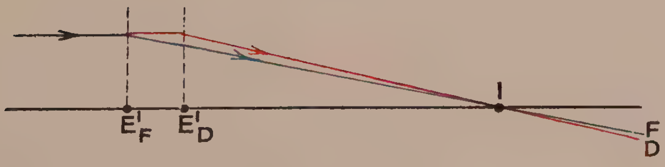

Fig. 42

[*Facing p. 52.*

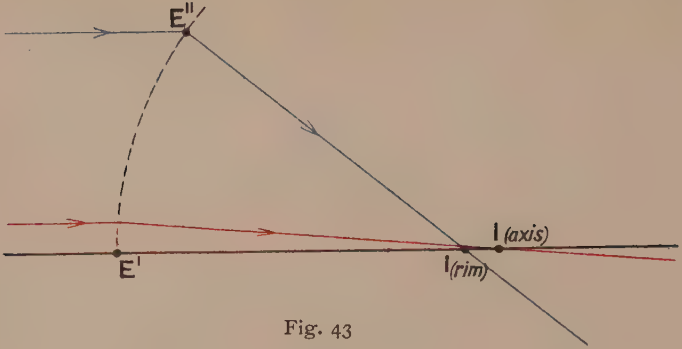

Fig. 43

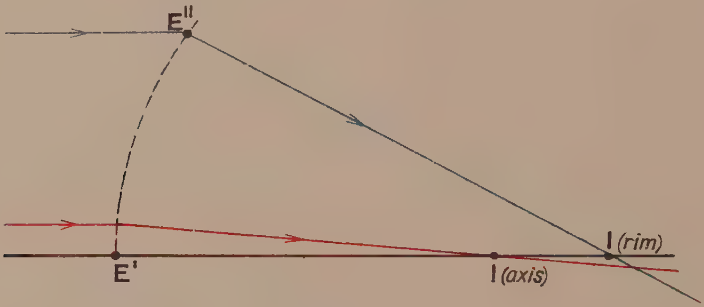

Fig. 44

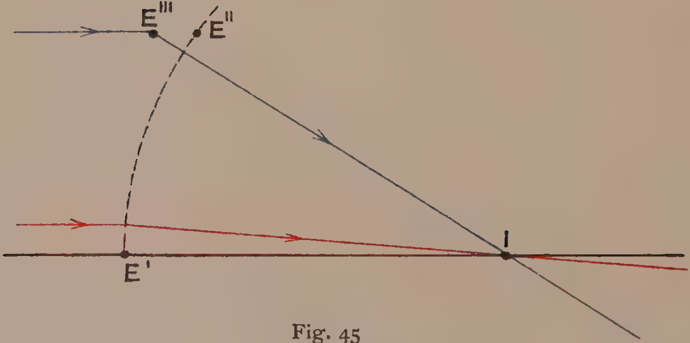

Fig. 45

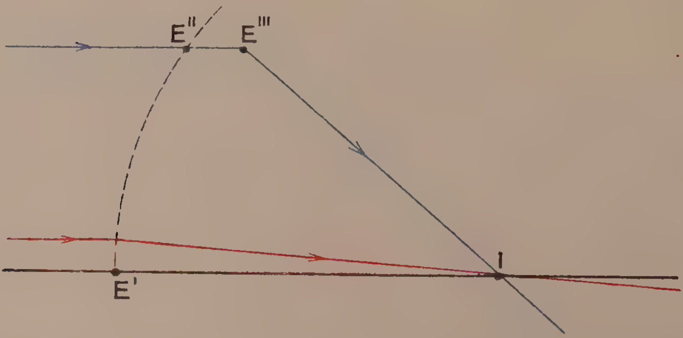

Fig. 46

and the *relative aperture* from the formula

$$(67)\qquad \frac{h}{f} = 1 : \frac{f}{h},$$

where $\frac{f}{h}$ is a numeral indicating the number of times the semi-aperture is contained in the focal length. In this way there is obtained a measure of the illumination which increases as the value $\frac{f}{h}$ decreases. The illumination varies inversely as the square of this value. The relationship of the semi angular aperture to the relative aperture is expressed by the equation

$$(68)\qquad 1 : \frac{f}{h} = \sin\beta.$$

The D ray parallel to the axis and incident at a height h should now be traced through the system, thus enabling the values of $v_{(\mathrm{D\ ray})}$ and $e'_{(\mathrm{D\ ray})}$ for this rim ray to be obtained. If the back focal length for the axial D ray is $V_2I_{(\mathrm{Axis})}$ and the back focal length for the D rim ray $V_2I_{(\mathrm{Rim})}$, then the difference of these two values is the *spherical aberration*. If the rim ray cuts the axis too soon, then the difference $V_2I_{(\mathrm{Axis})} - V_2I_{(\mathrm{Rim})}$ is plus. The focal length of the rim ray is in that case shorter than that of the axial ray and the spherical aberration is *uncorrected* (fig. 43, p. 53). In the opposite case it is *over-corrected* (fig. 44, p. 53). The principal points for rays incident at various heights should lie upon a sphere having its centre at the focus and its radius equal to the true focal length. The projections on the axis of these ideally situated principal points should therefore lie at a distance $2f\sin^2\frac{\beta}{2}$ behind the principal point of the axial rays when measured in the direction of the incident light. The distance of such a point of projection from the vertex of the last refracting surface is accordingly

$$(69)\qquad E'V_2 + 2f\sin^2\frac{\beta}{2} = e' + 2f\sin^2\frac{\beta}{2}.$$

There may now be obtained for the D rim ray the projection of its actual principal point E‴ (figs. 45, p. 53, and 46, p. 53). The distance of this projected principal point from the intersection of the rim ray with the axis is $\frac{h}{\tan\beta}$, and its distance from the vertex of the last surface is accordingly

$$(70)\qquad V_2I - \frac{h}{\tan\beta} = v - \frac{h}{\tan\beta}.$$

The difference between the projected principal and ideal points is a measure of *the distortion.* When its value,

$$(71)\qquad E'V_2 + 2f\sin^2\frac{\beta}{2} - V_2I + \frac{h}{\tan\beta},\text{ is positive,}$$

the distortion error is *uncorrected* (fig. 45), and in the reverse case it is *over-corrected* (as indicated in fig. 46).

That a displacement of the second principal point of the rim rays from its ideal position causes distortion of the image can easily be demonstrated. By tracing the passage through the system of an incident beam parallel to the axis and of a number of principal rays of different obliquities, it will be found that if the principal points of the rim rays lie upon a sphere having its centre at the focus and its radius equal to the true focal length, then all principal rays, even those that make a considerable angle with the axis, will emerge from the system as if they proceeded from the second principal point, and the image will accordingly be undistorted. If, however, the principal points of the rim rays do not lie upon this sphere, then the principal rays that make a considerable angle with the axis will appear to proceed from a point other than the second principal point; that is to say, the principal rays that are inclined at a considerable angle to the axis will not proceed from the same point as the rays that make a small angle, and the image will accordingly be distorted.

If, on tracing a rim ray of a second colour (corresponding with the F line), it is found that the back focal length, ${}_FV_2I$, is different from that of the D ray, namely, ${}_DV_2I$, there will exist a chromatic aberration of the rim ray which is *uncorrected*, as in fig. 47, p. 54, when the value ${}_DV_2I - {}_FV_2I$ is positive, and *over-corrected* when negative, as in fig. 48, p. 54.

If this particular error is eliminated simultaneously with the chromatic aberration of the axial ray, the system satisfies the

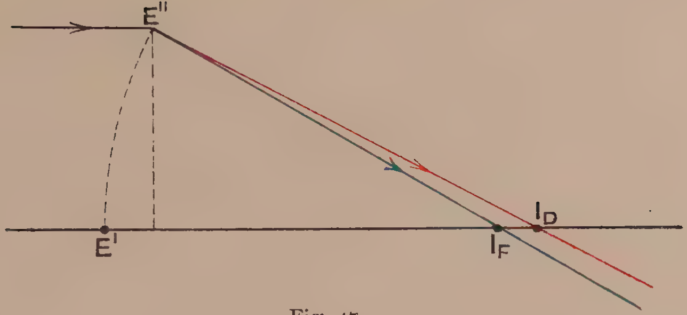

Fig. 47

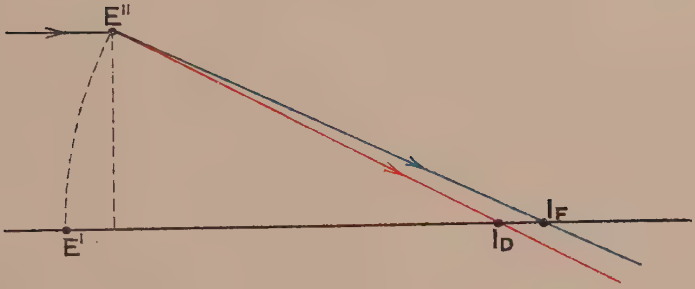

Fig. 48

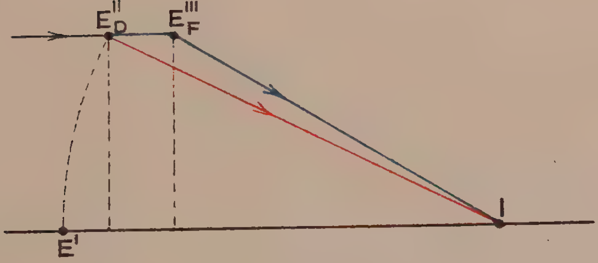

Fig. 49

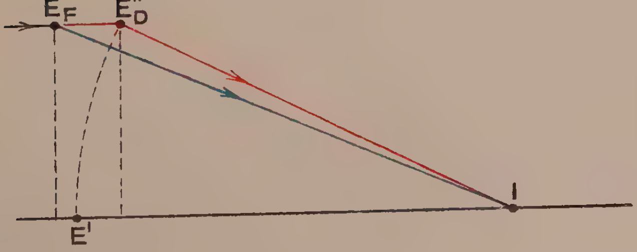

Fig. 50

[Facing p. 54.

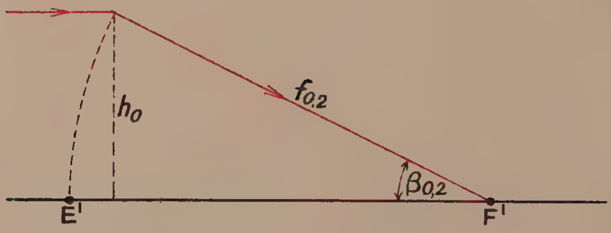

Fig. 51

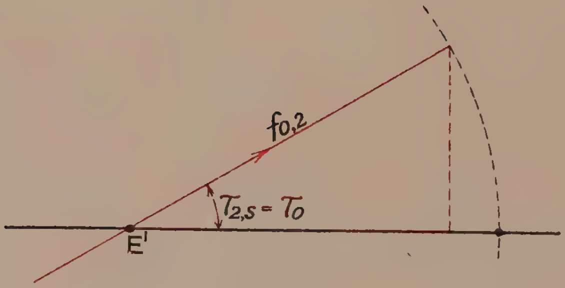

Fig. 52

Gauss condition. In such a case the spherical aberrations for both the F and D rays will be simultaneously removed.

When the rim ray of the first colour has the same focal point as the two axial D and F rays, the error in the back focal length of the corresponding rim ray of the second colour is called the *Gauss condition* aberration. This error is *uncorrected* when

$$(72) \qquad ({}_{D}V_2I - {}_{F}V_2I)_{\text{Axis}} - ({}_{D}V_2I - {}_{F}V_2I)_{\text{Rim}}$$

is positive, and vice versa.

If, finally, the tracing of the rim ray of the second colour, F, is found to give an incorrect position of the principal point, there will be present a chromatic difference of magnification for the rim ray which is *uncorrected*, as indicated in fig. 49, p. 54, when

$$(71) \qquad {}_{D}\left(E'V_2 + 2f\sin^2\frac{\beta}{2}\right) - {}_{F}\left(V_2I - \frac{h}{\tan\beta}\right)$$

is positive, and over-corrected in the reverse case (fig. 50, p. 54).

By tracing the rim ray for two colours there have been determined

1. The illumination of the system.
2. The spherical aberration.
3. The distortion error.
4. The chromatic aberration for a rim ray.
5. The chromatic difference of magnification for a rim ray.

In the case of an oblique incident beam four rays were chosen, namely: the principal, upper, lower, and side rays. For these four rays the respective aberrations must be determined. In the first instance, only those errors will be considered that arise in the case of monochromatic light corresponding with the D line; and, later, the errors arising from rays of another colour, corresponding with line F, will be referred to.

An oblique principal ray emerging from the system may appear to proceed from an axial point that is not identical with the second principal point. The emergent principal ray then no longer cuts the image surface at a height $f \sin \tau$ above the axis. If the point from which the principal ray appears to proceed lies nearer to the focal point, the principal ray meets the image surface lower down; when it is farther from the focus the principal ray meets the image surface at a greater height. This error causes *distortion*, which is said to be *uncorrected* when the point on the axis from which the emergent principal ray proceeds lies nearer the focus than the second principal point, and, conversely, *over-corrected* when farther from the focus.

When the upper, lower, and side rim rays forming an image point lying out of the axis intersect one another at a common point on the principal ray, the image point thus formed will be clearly defined. If this point of intersection is at the same time the point where the principal ray meets the image surface drawn through the focus with the principal point as centre and the true focal length as radius, then the image will be a correct one.‡

If the upper, lower, and side rim rays do not intersect in the same point, the image will be indistinct. The position of the image point is then regarded as being that plane normal to the optical axis in which the disk of light constituting the image is smallest.

If the two side rays intersect one another on the principal ray, and if the upper and lower rays also meet on the principal ray but in another point, then there is present *regular astigmatism*; if the upper and lower rays do not intersect upon the principal ray, then there is present *spherical aberration out of the axis*.

The two side rays lie symmetrically on either side of the plane containing the optical axis, the image point and the principal ray, and they always intersect one another in a point lying in this plane. If this point does not lie upon the principal ray, but is situated higher or lower, the error so produced is said to be one of *irregular astigmatism*.

In the case of an image point lying out of the axis the following errors may accordingly exist:—

1. Distortion.
2. Curvature of image.
3. Regular astigmatism.
4. Spherical aberration out of the axis.
5. Irregular astigmatism.
6. Combinations of these errors.

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‡ This is not strictly true, since the image surface corresponding with a plane object will be curved under the conditions named.—Ed.

The same errors may also exist in the case of rays of a second colour. Generally in practice it is sufficient to determine the errors only for one colour, if the points from which the power of the system has been obtained for the first colour—namely, the second principal point and the second focal point—are the same for the second colour.

A great simplification in the investigation of the extended images of a lens system arises from the fact that from the computation of the axial image points it is possible to see whether or not the most important conditions to be satisfied by the extended image are fulfilled to an appreciable extent. If, for example, on reckoning through the system, the focal and principal points for two colours for the axial and rim rays are found to be correctly situated, then the two complete images formed by the differently coloured rays will be equally large and undistorted, and will coincide.

In addition to this question of correct form and size, the computation of the axial image point indicates whether or not the clear definition of the image will diminish quickly from the centre of the field outwards. The want of definition of the images formed by an oblique beam is due to the asymmetric refraction of the three chosen oblique rays of the beam. If the rays are symmetrical the image will be clearly defined, and hence the best results will be obtained when the incident beam parallel to the axis passes through the system in the most symmetrical manner. This will be the case when a rim and an axial ray in their passage through the system are proportionally refracted, not only as regards their angles of deviation but also as regards their emergent heights. If these two rays intersect the axis in a common point (focus), and if, further, the ratio of the sine of the angle of incidence to the sine of the angle of emergence is the same for both the rim ray and the axial ray, then the rays pass through the lens system with the greatest possible symmetry. This condition, that the ratio of the sines of the angles of incidence and emergence shall be equal for both the rim and axial rays, is called the *sine condition.* This condition is satisfied when

$$(73) \qquad {}_{\text{Axis}}\left(\frac{\phi_0}{\phi'_{2s}}\right) = {}_{\text{Rim}}\left(\frac{\sin\,\phi_0}{\sin\,\phi'^{2s}}\right)$$

where in the case of the axial rays the arc is equal to the sine. The system is *uncorrected* as regards the sine condition when

$$(74) \qquad {}_{\text{Axis}}\left(\frac{\phi_0}{\phi'_{2s}}\right) - {}_{\text{Rim}}\left(\frac{\sin\phi_0}{\sin\phi'_{2s}}\right) \text{ is positive,}$$

and, conversely, it is *over-corrected* when the value is negative. The sine condition is fulfilled when the system has the same true focal length for any portion of the whole aperture, and it corresponds with the condition that the virtual points of emergence of the focal rays shall be correctly situated.

CHAPTER IV

The Computation of a Lens and the Discussion of its Image Aberrations

Preliminary Considerations

For the numerical determination of the aberrations of lenses, and later for the removal of these aberrations, in the case of a system of lenses, the trigonometrical method of computation will be adopted, since in this way there is obtained a picture of the paths of the rays that is true to a degree limited only by the choice of the order of the figures employed in the computation, and also, since those errors that cannot be removed are none the less always clearly expressed in such a way that it is easy to determine which of them is the greatest.

In order that the lenses selected for computation may be comparable, it is necessary, in the first instance, to fix their principal features; that is

1. The focal length, $f_{0,2}$.
2. The semi-aperture, h_0.
3. The semi angular field, τ_0.

1. The choice of the true focal length, that is, the distance of the focus from the principal point (for example, $f_{0,2} = 100$) determines the power of the lens, since the size of the image of an infinitely distant object of given angular diameter is directly proportional to the focal length. A definite focal length, $f_{0,2}$, is obtained by the computation of an axial ray.

2. When the focal length is fixed, the choice of the semi-diameter of an incident beam of light that is parallel to the axis,

equal to the semi-diameter of the working aperture of the lens, determines the illumination of the axial image point. If, for example, h_0 is 15, then the amount of light incident on the actual lens aperture is $h_0^2\pi$ ‡, as will be evident from fig. 51, p. 55, and

$$h_0 = f_{0,2} \sin \beta_{0,2},$$

where $\beta_{0,2}$ is the semi angular aperture, that is, the angle which the emergent rim ray makes with the axis. For a given focal length the illumination of the axial image point, therefore, varies directly as the square of the semi-aperture, or as the square of the sine of the semi angular aperture.§ It is necessary to decide upon the value of the semi-aperture for the computation of a rim ray.

When the focal length and the effective or daylight aperture of the lens are known, then all the rays from an object point in the axis that pass through the lens can be computed, when the form—that is, the radii and thickness—and the refractive index of the material of the lens are given.

3. The third factor is the angle, τ_0, which the extreme effective principal ray makes with the axis, fig. 52. The angular field is denoted by the angle, $2\tau_0$, which the two extreme effective principal rays make with one another. Further, by field of view is meant the area of the image surface thus produced, that is, $(f_{0,2} \sin \tau_0)^2 \times \pi$ ‡. The diameter of the field of view is $2 f_{0,2} \sin \tau_0$. The choice of the angle τ_0 is necessary for the computation of the rays proceeding from an object point out of the axis. From the data thus obtained it is now possible to compute all the rays from an object point situated out of the axis that proceed through the lens.

It is only possible to compare the images formed by two lenses when the latter are identical as regards the three abovementioned factors, namely focal length, semi-aperture, and angular field. If it is desired to compare any other lens of different dimensions with a given lens, it is necessary in the first instance to reduce the power, i.e. the focal length, of the former to equality with that of the given standard lens. All the other elements, namely, radii, thickness, daylight aperture, and diameter of field, must be multiplied by the same factor. If the adjusted semi-aperture is the same as that of the given standard lens, then the illuminations are the same. If the diameters of the fields of view are equal, then the two lenses may be directly compared. If, however, the adjusted semi-aperture is different from that of the standard, then the illumination will be different. It will be greater when the value of h_0 is greater, and vice versa.

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‡ The letter π, as here used, represents the ratio of the circumference to the diameter of the circle. It should not be confused with the letter π used to represent a specific angle, as described in the Notation for Angles, Chapter I.

§ Compare Chapter III, p. 30.

If h_0, $f_{0,2}$, $\beta_{0,2}$, and τ_0 represent the elements of the standard lens, and (h_0), $f_{0,2}$, $(\beta_{0,2})$, and (τ_0) the corresponding elements of the other lens after adjustment, then the equation

$$(75) \qquad h_0^2 x = (h_0)^2$$

expresses the relationship of the illuminations of the two lenses, where x indicates how many times the illumination of the standard lens is smaller or greater than that of the adjusted lens.

Similarly the equation

$$(76) \qquad \sin^2 \beta_{0,2}\, x = \sin^2(\beta_{0,2})$$

also expresses the relationship of the illuminations.

If, as a result of the adjustment, the diameter $(2 f_{0,2} \sin \tau_0)$ of the field of view becomes equal to that of the standard lens, then the angle τ_0 for the two lenses must be the same, since the two values of $f_{0,2}$ have been assumed equal. If the diameters are not the same, then the effective angular field of the adjusted lens is different from that of the standard, and its value τ_0 is obtained from the equation

$$(77) \qquad x \sin \tau_0 = \sin (\tau_0).$$

Having chosen the data, namely, focal length, semi-aperture, and angular field, that determine the performance of the lens to be constructed, it is then necessary to decide upon the form of the lens, which depends upon the distribution of the refraction over the two surfaces and upon the chosen thickness of the lens. The total deviation imparted by the lens to an axial ray is $\beta_{0,2}$, the value of which is obtained from the equation

$$(78) \qquad \sin \beta_{0,2} = \frac{h_0}{f_{0,2}}$$

(when the angular aperture is small this equation becomes $\beta_{0,2} = \frac{h_0}{f_{0,2}}$), where h_0 is the incidence height of the axial rays.

The following discussion deals with the case of the axial ray having any incidence height h_0, provided that it is small in relation to the focal length.

Assuming the focal length $f_{0,2} = 100$, the incident axial ray may be so chosen that the emergent ray of a definite colour, corresponding, for example, with the D line, is inclined to the axis at an angle of one second. Since both the sine and the arc of a second of angle may be regarded as being of the same value, 0·000004848137, the incidence height of the axial ray obtained from the formula $h_0 = f_{0,2} \sin 1''$ will be 0·0004848137. It would be inconvenient throughout the computations to make use of the values of $\beta_{0,2}$ and h_0 expressed as above in so many figures, and it is accordingly more convenient to multiply both sides of the equation by 206265, the reciprocal value of a second of arc.

Thus there is obtained the equation, (79)

$$\text{arc of } 1'' \times 206265 = \frac{\text{Incidence height of axial ray} \times 206265}{\text{Focal length}},$$

in which the arc of the semi angular aperture is to be made unity, and the incidence height 100, the equation then taking the form:

$$(80) \qquad 1 = \frac{100}{100}.$$

Accordingly the unit of angle adopted, namely the second, and the unit of incidence height, corresponding in the first instance with that of the focal length, should both be multiplied by 206265.

The semi angular aperture, $\beta_{0,2} = 1$, determined as above, represents the total deviation of an axial ray in its passage through the lens. This deviation is divided over the two lens surfaces. When the division of the refraction over the two surfaces is altered, their sum being kept constant and the focal length remaining unchanged, the form of the lens will vary in a manner that can be determined by computation.

Suppose the deviation at the first surface is δ_0 and at the second surface δ_2, and that in all cases $\beta_{0,2} = \delta_0 + \delta$. The distribution of the deviation over the two surfaces may then be varied as desired, for example:

$$(81) \qquad \begin{aligned} \delta_0 &= \delta_2 = \tfrac{1}{2}\beta_{0,2} = 0{\cdot}5, \text{ or} \\ \delta_0 &= 0; \; \delta_2 = \beta_{0,2} = 1{\cdot}0. \end{aligned}$$

Before proceeding to discuss these various cases, some consideration of the choice of the lens thicknesses is necessary. In general the smallest possible lens thickness should be chosen in order to avoid the use of unnecessarily large masses of glass. In the case of a convergent lens the thickness must be so chosen that the lens has the desired aperture and also sufficient rigidity. In the following computations a thickness $d_1 = 8$ mm. will be chosen.

When the lens is divergent the thickness must be such that the rigidity at the centre is sufficiently great. In certain cases of both converging and diverging lenses, greater thicknesses may be chosen in order to obtain definite effects. Further, it is necessary to select the value of the ratio of refraction, $\frac{n_{+1}}{n_{-1}}$, but since in most optical cases $n_{-1} = 1$, the value of n_{+1} alone remains to be chosen (for example, $n_{+1} = 1{\cdot}52964$). For the computation of a lens it is accordingly necessary to select a definite type of glass, and to choose for computation a ray of definite colour, such as the D ray.

Determination of the Lens Form

For the computation of a lens it is assumed, in accordance with the above remarks, that the values of h_0, d_1, and n_{+1} are known. To determine the form of the lens it is necessary to select the value of δ_0, from which is obtained the value, $\delta_2 = \beta_{0,2} - \delta_0$. The two radii of the lens, r_0 and r_2, can then be found by means of the formulæ

$$(82) \qquad r_0 = \frac{h_0(n_{+1} - 1)}{\delta_0 n_{+1}}, \text{ and}$$

$$(83) \qquad r_2 = \frac{(h_0 - d_1\delta_0)(n_{+1} - 1)}{-\delta_2 + \delta_0(n_{+1} - 1)}.$$

If r_0 or r_2 is positive the convex side of the corresponding surface is directed towards the incident light, and if negative, the concave side.

The correctness of the above formulæ may easily be established by means of the law of refraction, and by consider-

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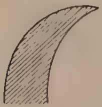
Convergent Lenses: $\beta_{0,2} = +1$.

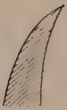
1. $\delta_0 = +2;\ \delta_2 = -1;\ \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = \frac{h_0(n_{+1} - 1)}{2n_{+1}},$$

$$r_2 = \frac{(h_0 - 2d_1)(n_{+1} - 1)}{2n_{+1} - 1}.$$

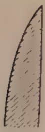
2. $\delta_0 = +1;\ \delta_2 = 0;\ \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = \frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = (h_0 - d_1).$$

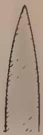
3. $\delta_0 = \frac{1}{n_{+1}};\ \delta_2 = \frac{n_{+1} - 1}{n_{+1}};\ \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = h_0(n_{+1} - 1),$$

$$r_2 = \infty.$$

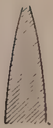
4. $\delta_0 = \delta_2 = 0{\cdot}5;\ \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = \frac{2h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -\frac{(2h_0 - d_1)(n_{+1} - 1)}{2 - n_{+1}}.$$

5. $\delta_0 + \delta_2 = \beta_{0,2} = +1;\ r_0 = -r_2.$



$$r_0 = -r_2 = h_0(n_{+1} - 1) \pm$$

$$\pm\sqrt{-\frac{h_0 d_1(n_{+1} - 1)^2}{n_{+1}} + h_0^2(n_{+1} - 1)^2}.$$

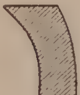
[*Continued on p. 66.*

ing the deviations of an axial ray at the two lens surfaces.

As examples, a number of cases will be considered in which the deviation is distributed in various ways over the two lens surfaces. The radii of the lenses thus determined are indicated in the following tables.

Divergent Lenses: $\beta_{0,2} = -1$.

1. $\delta_0 = -2;\ \delta_2 = +1;\ \beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = -\frac{h_0(n_{+1} - 1)}{2n_{+1}},$$

$$r_2 = -\frac{(h_0 + 2d_1)(n_{+1} - 1)}{2n_{+1} - 1}.$$

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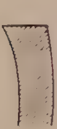
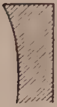
2. $\delta_0 = -1;\ \delta_2 = 0;\ \beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = -\frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -(h_0 + d_1).$$

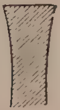
3. $\delta_0 = -\frac{1}{n_{+1}};\ \delta_2 = -\frac{n_{+1} - 1}{n_{+1}};\ \beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = -h_0(n_{+1} - 1),$$

$$r_2 = \infty.$$

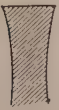
4. $\delta_0 = \delta_2 = -0{\cdot}5;\ \beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = -\frac{2h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = +\frac{(2h_0 + d_1)(n_{+1} - 1)}{2 - n_{+1}}.$$

5. $\delta_0 + \delta_2 = \beta_{0,2} = -1;\ r_0 = -r_2$.



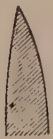
$$r_0 = -r_2 = -h_0(n_{+1} - 1) \pm$$

$$\pm\sqrt{+\frac{h_0 d_1 (n_{+1} - 1)^2}{n_{+1}} + h_0^2(n_{+1} - 1)^2}.$$

[*Continued on p. 67.*

Convergent Lenses (*Continued*)

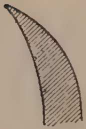
6. $\delta_0 = 0; \quad \delta_2 = +1; \quad \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = \infty,$$

$$r_2 = -h_0(n_{+1} - 1).$$

7. $\delta_0 = -1; \quad \delta_2 = +2; \quad \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = -\frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -\frac{(h_0 + d_1)(n_{+1} - 1)}{n_{+1} + 1}.$$

In the example chosen, where $n_{+1} = 1{\cdot}529\,64$, $h_0 = 100$, and $d_1 = 8$, the following results are obtained:—

Convergent Lenses: $\beta_{0,2} = +1.$

1a.‡ $\delta_0 = +2{\cdot}2; \quad \delta_2 = -1{\cdot}2.$

$$r_0 = \frac{100\,(0{\cdot}529\,64)}{2{\cdot}2 \times 1{\cdot}529\,64} = +15{\cdot}739.$$

$$r_2 = \frac{(100 - 2{\cdot}2 \times 8)\,0{\cdot}529\,64}{2{\cdot}2 \times 0{\cdot}529\,64 + 1{\cdot}2} = +18{\cdot}452.$$

1. $\delta_0 = +2{\cdot}0; \quad \delta_2 = -1{\cdot}0.$

$$r_0 = \frac{100\,(0{\cdot}529\,64)}{2 \times 1{\cdot}529\,64} = +17{\cdot}312\,5.$$

$$r_2 = \frac{(100 - 2 \times 8)\,0{\cdot}529\,64}{2 \times 1{\cdot}529\,64 - 1} = +21{\cdot}605.$$

2. $\delta_0 = +1{\cdot}0; \quad \delta_2 = 0{\cdot}0.$

$$r_0 = \frac{100\,(0{\cdot}529\,64)}{1{\cdot}529\,64} = +34{\cdot}625.$$

$$r_2 = 100 - 8 = +92{\cdot}000.$$

3. $\delta_0 = \frac{1}{n_{+1}}; \quad \delta_2 = \frac{n_{+1} - 1}{n_{+1}}.$

$$r_0 = 100 \times 0{\cdot}529\,64 = +52{\cdot}964.$$

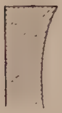
$$r_2 = \infty.$$

‡ The types 1a are included as being the limiting forms for the given example.

[*Continued on p. 68.*

Divergent Lenses (*Continued*)

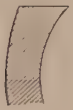
6. $\delta_0 = 0$; $\delta_2 = -1$; $\beta_{0,2} \doteq \delta_0 + \delta_2 = -1$.



$$r_0 = \infty,$$

$$r_2 = +h_0(n_{+1} - 1).$$

7. $\delta_0 = +1$; $\delta_2 = -2$; $\beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = +\frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = +\frac{(h_0 - d_1)(n_{+1} - 1)}{n_{+1} + 1}.$$

In the example chosen, where $n_{+1} = 1{\cdot}529\,64$, $h_0 = 100$, and $d_1 = 8$, the following results are obtained:—

Divergent Lenses: $\beta_{0,2} = -1$.

1a.‡ $\delta_0 = -2{\cdot}2$; $\delta_2 = +1{\cdot}2$.

$$r_0 = -\frac{100\,(0{\cdot}529\,64)}{2{\cdot}2 \times 1{\cdot}529\,64} = -15{\cdot}739.$$

$$r_2 = -\frac{(100 + 2{\cdot}2 \times 8)\,(0{\cdot}529\,64)}{2{\cdot}2 \times 0{\cdot}529\,64 + 1{\cdot}2} = -26{\cdot}334.\S$$

1. $\delta_0 = -2{\cdot}0$; $\delta_2 = +1{\cdot}0$.

$$r_0 = -\frac{100\,(0{\cdot}529\,64)}{2 \times 1{\cdot}529\,64} = -17{\cdot}312\,5.$$

$$r_2 = -\frac{(100 + 2 \times 8)\,(0{\cdot}529\,64)}{2 \times 1{\cdot}529\,64 - 1} = -29{\cdot}835.\S$$

2. $\delta_0 = -1{\cdot}0$; $\delta_2 = 0{\cdot}0$.

$$r_0 = -\frac{100\,(0{\cdot}529\,64)}{1{\cdot}529\,64} = -34{\cdot}625.$$

$$r_2 = -(100 + 8) = -108{\cdot}00.\S$$

3. $\delta_0 = -\dfrac{1}{n_{+1}}$; $\delta_2 = -\dfrac{n_{+1} - 1}{n_{+1}}$.

$$r_0 = -100 \times 0{\cdot}529\,64 = -52{\cdot}964.$$

$$r_2 = \infty.$$

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‡ The types 1a are included as being the limiting forms for the given example.

§ A number of important mistakes in the original divergent-lens results have been corrected.—Ed.

[*Continued on p. 69.*

Convergent Lenses (*Continued*)

4. $\delta_0 = \delta_2 = 0{\cdot}5.$

$$r_0 = \frac{2 \times 100\,(0{\cdot}529\,64)}{1{\cdot}529\,64} = +69{\cdot}250.$$

$$r_2 = -\frac{(2 \times 100 - 8)\,(0{\cdot}529\,64)}{2 - 1{\cdot}529\,64} = -216{\cdot}195.$$

5. $\delta_0 + \delta_2 = 1; \quad r_0 = -r_2.$

$$r_0 = -r_2 = 100 \times 0{\cdot}529\,64 \pm$$

$$\pm\sqrt{-\frac{100 \times 8\,(0{\cdot}529\,64)^2}{1{\cdot}529\,64} + 100^2(0{\cdot}529\,64)^2} = +104{\cdot}524.$$

6. $\delta_0 = 0; \quad \delta_2 = +1.$

$$r_0 = \infty.$$

$$r_2 = -100 \times 0{\cdot}529\,64 = -52{\cdot}964.$$

7. $\delta_0 = -1; \quad \delta_2 = +2.$

$$r_0 = -\frac{100\,(0{\cdot}529\,64)}{1{\cdot}529\,64} = -34{\cdot}625.$$

$$r_2 = -\frac{(100 + 8)\,(0{\cdot}529\,64)}{2{\cdot}529\,64} = -22{\cdot}612.$$

7a.‡ $\delta_0 = -1{\cdot}2; \quad \delta_2 = +2{\cdot}2.$

$$r_0 = -\frac{100\,(0{\cdot}529\,64)}{1{\cdot}835\,57} = -28{\cdot}854.$$

$$r_2 = -\frac{(100 + 9{\cdot}6)\,(0{\cdot}529\,64)}{2{\cdot}835\,57} = -20{\cdot}472.$$

‡ The types 7a are included as being limiting forms.

When determining the form of the lens it is necessary to observe what is the size of the effective aperture. On the one hand, the maximum aperture of the lens is limited by its form; rays that lie at a greater distance from the axis than the radius of the maximum aperture no longer fall upon the lens. On the other hand, certain rays that fall upon the lens may be totally reflected from one or other of the lens surfaces and thus be prevented from contributing to the formation of an image by the lens.

Divergent Lenses (*Continued*)

4. $\delta_0 = \delta_2 = -0 \cdot 5.$

$$r_0 = -\frac{2 \times 100 (0 \cdot 529\,64)}{1 \cdot 529\,64} = -69 \cdot 250.$$

$$r_2 = +\frac{(2 \times 100 + 8)(0 \cdot 529\,64)}{2 - 1 \cdot 529\,64} = +234 \cdot 214.‡$$

5. $\delta_0 + \delta_2 = -1; \quad r_0 = -r_2.$

$$r_0 = -r_2 = -100 \times 0 \cdot 529\,64 \pm$$

$$\pm\sqrt{+\frac{100 \times 8 (0 \cdot 529\,64)^2}{1 \cdot 529\,64} + 100^2(0 \cdot 529\,64)^2} = -107 \cdot 289.‡$$

6. $\delta_0 = 0; \quad \delta_2 = -1.$

$$r_0 = \infty.$$

$$r_2 = 100 \times 0 \cdot 529\,64 = +52 \cdot 964.$$

7. $\delta_0 = +1; \quad \delta_2 = -2.$

$$r_0 = \frac{100 (0 \cdot 529\,64)}{1 \cdot 529\,64} = +34 \cdot 625.$$

$$r_2 = \frac{(100 - 8)(0 \cdot 529\,64)}{2 \cdot 529\,64} = +19 \cdot 262.‡$$

7a.§ $\delta_0 = +1 \cdot 2; \quad \delta_2 = -2 \cdot 2.$

$$r_0 = \frac{100 (0 \cdot 529\,64)}{1 \cdot 835\,57} = +28 \cdot 854.$$

$$r_2 = \frac{(100 - 9 \cdot 6)(0 \cdot 529\,64)}{2 \cdot 835\,57} = +16 \cdot 885.‡$$

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‡ A number of important mistakes in the original divergent-lens results have been corrected.—Ed.

§ The types 7a are included as being limiting forms.

(*a*) *Maximum Aperture of a Lens*

In the case of a convergent lens the radius of the maximum aperture may be obtained from the following formula (84):—

$$h_0 = \frac{2}{+r_2 - r_0 + d_1}\sqrt{\frac{d_1}{2}\left(-r_0 + r_2 + \frac{d_1}{2}\right)\left(+r_0 - \frac{d_1}{2}\right)\left(+r_2 + \frac{d_1}{2}\right)},$$

which can easily be deduced from fig. 53. When $d_1 = +2r_0$ or $d_1 = -2r_2$, it follows from the equation that $h_0 = 0$. When

d_1 is greater than $2r_0$ or $-2r_2$ the equation gives imaginary results; the lens, instead of having a sharp edge, will then be in the form of a cylinder having its axis coincident with that of the lens surfaces, and having its maximum semi-aperture equal to the smaller radius of curvature. In many cases it is more convenient to substitute a conical surface for the cylindrical surface.

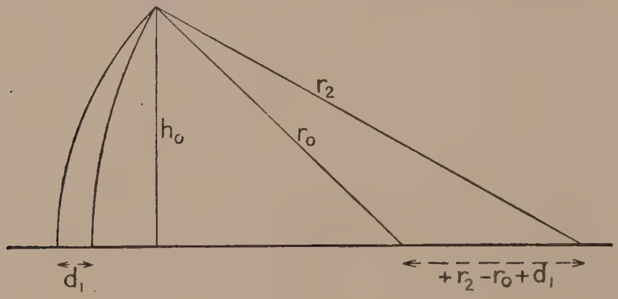

Fig. 53

In certain special cases the general formula for h_0 becomes simplified, for example, when

$$(85) \qquad r_0 = \infty,$$

$$h_0 = 2\sqrt{\frac{d_1}{2}\left(-r_2 - \frac{d_1}{2}\right)};$$

and further, when $r_2 = \infty$,

$$(86) \qquad h_0 = 2\sqrt{\frac{d_1}{2}\left(+r_0 - \frac{d_1}{2}\right)}.$$

From these formulæ it will be seen that in the examples previously selected (p. 66), the maximum apertures h_0 are as follows:—

$r_0 =$	15·739	17·312	34·625	52·964	69·250	104·524	∞	34·625	28·854
$r_2 =$	18·452	21·605	92·000	∞	216·195	104·524	52·964	22·612	20·472
$d_1 =$	8	8	8	8	8	8	8	8	8
$h_0 =$	15·705	17·300	25·990	27·990	28·460	28·639	27·990	21·444	19·987

In the case of lenses of equal power and thickness the equi-convex lens has the largest aperture, and the aperture diminishes as the form alters in either direction. Further, the thickness of the lens affects the maximum aperture. If the power and form of the lens are kept constant, and the thickness is altered from 8 to 48, 88, and 138·5, the results will be as follows:—

$r_0 =$	+69·25	+69·25	+69·25	+69·25
$r_2 =$	−216·195	−171·153	−126·115	−69·25
$d_1 =$	8	48	88	138·5
$h_0 =$	28·46	61·136	69·22	69·25

It will be seen, as was to be expected, that, as the lens thickness increases, the maximum aperture increases. In divergent lenses the radius of the maximum aperture corresponds with the smaller radius of curvature.

(*b*) *The Effect of Total Reflection in reducing the Maximum Aperture*

In Chapter I, p. 4, it is stated that when a ray passes from a dense medium into a less dense one, total reflection of the ray occurs when the angle of incidence is equal to or greater than the value obtained from the equation

$$(87) \qquad \sin \phi_{2s} = \frac{n_{2s+1}}{n_{2s-1}}.$$

For example, in a crown glass having a refractive index for the H line of 1·547, the incidence angle must be greater than 40° 16′ 18″, and in a flint glass having a refractive index 1·671 for the H line, greater than 36° 45′ 31″.[‡]

In the trigonometrical computation of rays through a lens the possibility of total reflection may be determined by observing whether or not the sine of the angle of refraction at the last surface has reached the value unity.

Choice of the Rays to be reckoned through the System

Having determined the radii of curvature of the lens surfaces, the values of h_0, d_1, n_{+1}, r_0, and r_2 are known, and it is now possible to trace trigonometrically the paths of all rays proceeding to the lens from an object point *on the axis* situated at infinity. When in addition the angle τ_0 is given, it is possible to trace trigonometrically the paths of all rays proceeding to the lens from a second object at an infinite distance but situated *out of the axis*. It is not practicable to compute the paths of all rays, and it is important therefore to make such a choice that by the computation of the smallest possible number of rays a clear representation of the qualities of the image will be obtained.

[‡] That is, when the other medium is air, and the value of n_{2s+1} is accordingly unity.—Ed.

For this purpose there will be chosen two, and in some cases three, of the rays that proceed from the object point in the axis, namely:

1. One parallel to the axis and situated close to it, called the *paraxial ray*.
2. One parallel to the axis but incident at the edge of the effective aperture, called the *rim ray*.

and, in certain cases,

3. A parallel ray situated at a distance from the axis equal to two-thirds of the radius of the effective aperture, called the *two-thirds rim ray*.

Owing to the dispersion that occurs at the first refracting surface it is necessary to trace the rays through the system with respect to two colours.

The object point situated out of the axis is determined by two angles. One is the angle τ_0, which the principal ray proceeding from the object point makes with the optical axis; it is measured in the plane containing the optical axis and the object point. The second is the angle between the vertical and the projection of the incident principal ray on a plane normal to the axis, and passing through the first principal point. It is reckoned from zero in a right-handed direction through 90°, 180°, 270°, to 360°. Thus, for example, this angle π_0‡ is equal to 180° when the image point is vertically under the axis.

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‡ The plane in which the angle π_0 is measured can assume another position, while remaining normal to the optical axis; for example, it may correspond with a plane passing through the centre, in which case the above-mentioned definition of the angle π_0 corresponds with that adopted by Von Seidel in the development of his trigonometrical formulæ. (See p. 11.)

In the case of an image point lying out of the axis, the following rays are chosen for computation:—

1. The principal ray, that is, the central ray, of the oblique bundle.
2. The upper, and
3. The lower rim ray, that is to say, two peripheral rays of the incident beam parallel to the principal ray and lying in the same vertical plane, one being above the axis of the beam and the other below.
4. The ray which is also parallel to the principal ray and lies on the periphery of the beam but which is oriented through 90°, say towards the right, with respect to the upper or lower ray. There is a corresponding ray lying symmetrically towards the left, but it is only necessary to compute one of these two rays.

The planes in which the refraction of the last-mentioned ray takes place change from surface to surface. In certain cases it is necessary to trace additional two-third rim rays corresponding with those mentioned under paragraphs 2, 3, and 4. It is generally unnecessary to consider rays of a second colour in the case of objects out of the axis, more especially when the different coloured rays have the same focus and equal focal lengths. When the field of view is large, it is necessary to trace the rays proceeding to a third image point lying in an intermediate position out of the axis in order to determine zonal aberrations.

Determination of the Limits of Uncertainty in the Computations

Before commencing the trigonometrical computations it is necessary to decide to what number of significant figures the calculations must be made. The number of significant figures employed should not be more than is actually necessary to give results of the desired accuracy. If, for example, five-figure logarithms are used, it is necessary to consider what is the degree of accuracy attainable in the results.

Numbers employed in the calculation must contain such a number of digits that the negligible portion corresponds to not more than five units in the sixth place of the logarithm. For example, in the case of the logarithm 0·18459, the corresponding number should contain six figures, namely 1·52964. For, if only the first five figures, 1·5296, were taken, the logarithm would be uncertain to 12·1 units in the sixth place, since the logarithm of 1·5296 = 0·1845779.

In this connection it will be clear that for all numbers smaller than 43455, in which one unit in the fifth place corresponds with a difference of 100 when using seven-figure logarithms, and with a difference of one when using five-figure logarithms, it is necessary to write down the number to six places. For all greater numbers the use of six figures serves no useful purpose, the first five places being sufficient.

It is still more important, when dealing with angles, to use the correct number of places, and it is necessary in each case to decide with what accuracy the angles must be expressed: whether an accuracy of ten seconds or of single seconds is sufficient, or if it is necessary to include also tenths of a second. This decision must be made in the case of rim rays, but it is not necessary for axial rays in the computation of which the arc is used instead of the angle.

There will first be considered how many seconds of the total angle of deviation, that is, of $\beta_{0,2}$, correspond with one unit in the fifth place; for example, suppose—

$$\sin \beta_{0,2} = \frac{h_0}{f_{0,2}} = \frac{15}{100},$$

$$\text{then} \quad \log \sin \beta_{0,2} = 9{\cdot}17609,$$

$$\text{and} \quad \beta_{0,2} = 8^\circ\, 37'\, 37'' = 31057''.$$

One second of this angle corresponds with 13·88 units in the sixth place of the logarithm, and therefore five units in the sixth place of the logarithm correspond with 0·36″.

Or, if

$$\sin \beta_{0,2} = \frac{10}{100},$$

$$\text{then} \quad \log \sin \beta_{0,2} = 9{\cdot}00000,$$

$$\text{and} \quad \beta_{0,2} = 5^\circ\, 44'\, 21{\cdot}0'' = 20661'',$$

in which case five units in the sixth place correspond with 0·24″.

Or, lastly, if

$$\sin \beta_{0,2} = \frac{6{\cdot}6667}{100},$$

$$\text{then} \quad \log \sin \beta_{0,2} = 8{\cdot}82391,$$

$$\text{and} \quad \beta_{0,2} = 3^\circ\, 49'\, 21{\cdot}2'' = 13761{\cdot}2'',$$

and five units in the sixth place correspond with 0·16″. Accordingly, in the first case it is necessary to work to half-seconds, and in the two last cases to tenths of a second. Thus the value of the angle that ensures a result accurate to five units in the sixth place is determined when the angle in question, namely, the total deviation of the lens, is the greatest angle that occurs. This accuracy may, however, be less when individual angles of refraction are greater than the total angle of deviation suffered by the ray. Accordingly, it is necessary to decide what value of the angle corresponds with five units in the sixth place in the case of the largest angle that occurs in the calculation.

In the cases already considered, suppose the greatest angles to be 12° 38′ 44″, 8° 20′ 31·1″, and 5° 32′ 9·3″ respectively. The corresponding logarithmic sines are 9·34028, 9·16175, and 8·98439, and five units in the sixth place then correspond with 0·5″, 0·35″, and 0·23″, respectively. No angular value need therefore be specified to a greater degree of accuracy than is represented by the above errors. But it should be noted that the results are only true to this order—that is, in the above-mentioned cases—to 1″ and 0·5″ respectively.

If the largest angle that occurred in the calculation were 30°, then the discarding of five units in the sixth place would involve 1·4″, and the uncertainty in the results would accordingly be 3″. If results certain to 1″ are desired, it is necessary in the case of so large an angle to use six-figure logarithms.

Formulæ for the Trigonometrical Computation through a Lens of the Paths of Axial and Rim Rays

For reckoning the paths of rays through a lens, trigonometrical formulæ that can easily be evolved will be used. In passing from one refraction to another, longitudinal dimensions

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(a) *For any Incident Ray that is parallel to the Axis*

I. When no Radius is very large:—

1. $\sin \phi_0 = \dfrac{h_0}{+r_0}.$

2. $\sin \phi_0' = \dfrac{n_{-1}}{n_{+1}} \sin \phi_0.$

3. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$

4. $a_0' - r_0 = +r_0 \dfrac{\sin \phi_0'}{\sin \beta_{0,0}}.$

5. $\sin \phi_2 = \dfrac{a_0' - r_0 + r_0 - r_2 - d_1) \sin \beta_{0,0}}{+ r_2}.$

6. $\sin \phi_2' = \dfrac{n_{+1}}{n_{+3}} \sin \phi_2.$

7. $\delta_2 = \phi_2 - \phi_2'.$

8. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$

9. $a_2' - r_2 = \dfrac{+r_2 \sin \phi_2'}{\sin \beta_{0,2}}.$

10. $f_{0,2} = \dfrac{h_0}{\tan \beta_{0,2}}.$

11. $e_{0,2} = a_2' - f_{0,2}.$

in the direction of the axis (abscissæ), or dimensions at right angles (ordinates or heights), may be employed. In general, the formulæ involving abscissæ are simpler, but when the radii of the lenses are great, such formulæ give less accurate results. It is then necessary to use the formulæ involving heights. When the first or last surface is plane, the height formulæ are somewhat simpler to use. The various formulæ are given in the following tables.

(*a*) *For any Incident Ray that is parallel to the Axis*

II. When one Radius is large‡:—

1. $\sin \phi_0 = \dfrac{h_0}{+r_0}.$
2. $\sin \phi_0' = \dfrac{n_{-1}}{n_{+1}} \sin \phi_0.$
3. $\eta_0 = \phi_0.$
4. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$
5. $a_0' = \dfrac{h_0}{\tan \beta_{0,0}} + 2r_0 \sin^2 \dfrac{\eta_0}{2}.$
6. $\sin \phi_2 = \dfrac{(a_0' - r_2 - d_1) \sin \beta_{0,0}}{+r_2}.$
7. $\sin \phi_2' = \dfrac{n_{+1}}{n_{+3}} \sin \phi_2.$
8. $\delta_2 = \phi_2 - \phi_2'.$
9. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$
10. $\eta_2 = \phi_2 + \beta_{0,0}.$
11. $h_2 = \left(a_0' - d_1 - 2r_2 \sin^2 \dfrac{\eta_2}{2}\right) \tan \beta_{0,0}.$
12. $a_2' = \dfrac{h_2}{\tan \beta_{0,2}} + 2r_2 \sin^2 \dfrac{\eta_2}{2}.$
13. $f_{0,2} = \dfrac{h_0}{\tan \beta_{0,2}}.$
14. $e_{0,2} = a_2' - f_{0,2}.$

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‡ The "height" formulæ must also be used when it is necessary to know the heights at which a ray meets the surfaces, as, for example, in order to determine the thicknesses of the lenses.

III. When the First Surface is plane:—

1. $\sin \phi_2 = \dfrac{h_0}{+r_2}.$

2. $\sin \phi_2' = \dfrac{n_{+1}}{n_{+3}} \sin \phi_2.$

3. $\delta_2 = \phi_2 - \phi_2'.$

4. $\beta_{0,2} = \delta_2.$

5. $a_2' - r_2 = \dfrac{+r_2 \sin \phi_2'}{\sin \beta_{0,2}}.$

6. $f_{0,2} = \dfrac{h_0}{\tan \beta_{0,2}}.$

7. $e_{0,2} = a_2' - f_{0,2}.$

If the ray parallel to the axis is incident at a point very near to it, the angles involved are so small that the arc may be substituted for the sine, in which case the above formulæ become somewhat simplified.

(*b*) *For a Paraxial Ray*

I. When no Radius is very large:—

1. $\phi_0 = \dfrac{h_0}{+r_0}.$

2. $\phi_0' = \dfrac{n_{-1}}{n_{+1}} \phi_0.$

3. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0 .$

4. $a_0' = +r_0 + \dfrac{r_0 \phi_0'}{\beta_{0,0}}.$

[*Continued on p. 80.*

IV. When the Second Surface is plane:—

1. $\sin \phi_0 = \frac{h_0}{+r_0}.$

2. $\sin \phi_0' = \frac{n_{-1}}{n_{+1}} \sin \phi_0.$

3. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$

4. $a_0' - r_0 = +r_0 \frac{\sin \phi_0'}{\sin \beta_{0,0}}.$

5. $\sin \phi_2 = -\sin \beta_{0,0}.$

6. $\sin \phi_2' = \frac{n_{+1}}{n_{+3}} \sin \phi_2.$

7. $\delta_2 = \phi_2 - \phi_2'.$

8. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$

9. $a_2' = (a_0' - d_1) \frac{\tan \beta_{0,0}}{\tan \beta_{0,2}}.$

10. $f_{0,2} = \frac{h_0}{\tan \beta_{0,2}}.$

11. $e_{0,2} = a_2' - f_{0,2}.$

(b) *For a Paraxial Ray*

II. When the Radii are large‡:—

1. $\phi_0 = \frac{h_0}{+r_0}.$

2. $\phi_0' = \frac{n_{-1}}{n_{+1}} \phi_0.$

3. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$

4. $a_0' = \frac{h_0}{\beta_{0,0}}.$

‡ The "height" formulæ must also be used when it is necessary to know the heights at which a ray meets the surfaces, for example, in order to determine the thicknesses of the lenses.

[*Continued on p. 81.*

When no Radius is very large (*Continued*):—

5. $\phi_2 = \dfrac{(a_0' - r_2 - d_1)\,\beta_{0,0}}{+r_2}.$

6. $\phi_2' = \dfrac{n_{+1}}{n_{+3}}\,\phi_2.$

7. $\delta_2 = \phi_2 - \phi_2'.$

8. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$

9. $a_2' = +r_2 + r_2\dfrac{\phi_2'}{\beta}.$

10. $f_{0,2} = \dfrac{h_0}{\beta_{0,2}}.$

11. $e_{0,2} = a_2' - f_{0,2}.$

III. When the First Surface is plane:—

1. $\phi_2 = \dfrac{h_0}{+r_2}.$

2. $\phi_2' = \dfrac{n_{+1}}{n_{+3}}\,\phi_2.$

3. $\delta_2 = \phi_2 - \phi_2'.$

4. $\beta_{0,2} = \delta_2.$

5. $a_2' = +r_2 + r_2\dfrac{\phi_2'}{\beta_{0,2}}.$

6. $f_{0,2} = \dfrac{h_0}{\beta_{0,2}}.$

7. $e_{0,2} = a_2' - f_{0,2}.$

When the Radii are large (*Continued*):—

5. $\phi_2 = \frac{(a_0' - r_2 - d_1)\,\beta_{0,0}}{+r_2}.$

6. $\phi_2' = \frac{n_{+1}}{n_{+3}}\,\phi_2.$

7. $\delta_2 = \phi_2 - \phi_2'.$

8. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$

9. $h_2 = (a_0' - d_1)\,\beta_{0,0}.$

10. $a_2' = \frac{h_2}{\beta_{0,2}}.$

11. $f_{0,2} = \frac{h_0}{\beta_{0,2}}.$

12. $e_{0,2} = a_2' - f_{0,2}.$

IV. When the Second Surface is plane:—

1. $\phi_0 = \frac{h_0}{+r_0}.$

2. $\phi_0' = \frac{n_{-1}}{n_{+1}}\,\phi_0.$

3. $\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$

4. $a_0' = +r_0 + \frac{r_0\phi_0'}{\beta_{0,0}}.$

5. $\phi_2 = -\beta_{0,0}.$

6. $\phi_2' = \frac{n_{+1}}{n_{+3}}\,\phi_2.$

7. $\delta_2 = \phi_2 - \phi_2'.$

8. $\beta_{0,2} = \delta_2 + \beta_{0,0}.$

9. $a_2' = (a_0' - d_1)\,\frac{\beta_{0,0}}{\beta_{0,2}}.$

10. $f_{0,2} = \frac{h_0}{\beta_{0,2}}.$

11. $e_{0,2} = a_2' - f_{0,2}.$

Examples of the Computation of Axial and Rim Rays

As examples, three convergent lenses will be chosen, namely: one in which the deviations at the two faces are equal; one in which one surface is of long radius; and, lastly, one having one face plane.

I. Convergent lens of crown-glass material:

Refractive indices, $n_{-1} = n_{+3} = 1$,
and ${}_{D}n_{+1} = 1{\cdot}52964$,
and ${}_{F}n_{+1} = 1{\cdot}53639$.

Total deviation of the lens, $\beta_{0,2} = 1$.
Deviations at the two surfaces, δ_0 and δ_2, each $= 0{\cdot}5$.
Lens thickness, $d_1 = 8$.

As in the example 4, p. 68, the radii will accordingly be $r_0 = +69{\cdot}250$, and $r_2 = -216{\cdot}195$.

In the first instance, the path of an axial ray corresponding with the D line will be traced through the lens, and the formulæ, Table I, in which no values of incidence heights are determined, will be used, since in the case chosen neither of the lens surfaces is of large radius.

For purposes of comparison, the results obtained from the height formulæ are indicated alongside those obtained from the other formulæ which do not involve heights. In certain instances the greater accuracy of the height formulæ is evident.

In addition to the above-mentioned axial ray, there will be computed the axial ray corresponding with the F line, and, finally, in order to find the first principal point of the lens, an axial ray corresponding with the D line will be traced through the lens in the opposite direction. In such a case it is necessary to take $r_0 = +216{\cdot}195$ and $r_2 = -69{\cdot}250$.

I. Convergent lens of crown glass having the refractive indices $n_{-1} = n_{+3} = 1$, ${}_{D}n_{+1} = 1·529\,64$, and ${}_{F}n_{+1} = 1·536\,39$. Let the total deviation of the lens $\beta_{0,2} = 1$, and the deviations at the two surfaces δ_0 and δ_2 each $= 0·5$. Let the thickness of the lens $d_1 = 8$.

Crown: ${}_{D}n_{+1} = 1·529\,64$ $\log {}_{D}n_{+1} = 0·184\,59$ $d_1 = 8;$ $r_0 = +69·250$ $\log r_0 = 1·840\,42$

${}_{F}n_{+1} = 1·536\,39$ $\log {}_{F}n_{+1} = 0·186\,50$ $r_2 = -216·195$ $\log r_2 = 2·334\,85\,n$

Paraxial Rays.

D.				F.		Lens Reversed: D.	
$\log h_0 =$	2·000 00			$\log h_0 =$	2·000 00	$\log h_2 =$	2·000 00
$\log (1 : r_0) =$	8·159 58			$\log (1 : r_0) =$	8·159 58	$\log (1 : r_2) =$	7·665 15
$\log \phi_0 =$	0·159 58			$\log \phi_0 =$	0·159 58	$\log \phi_2 =$	9·665 15
$\log (n_{-1} : n_{+1}) =$	9·815 41			$\log (n_{-1} : n_{+1}) =$	9·813 50	$\log (n_{+3} : n_{+1}) =$	9·815 41
$\log \phi_0' =$	9·974 99			$\log \phi_0' =$	9·973 08	$\log \phi_2' =$	9·480 56
$\phi_0 =$	1·444 04			$\phi_0 =$	1·444 04	$\phi_2 =$	0·462 54
$-\phi_0' =$	−0·944 04			$-\phi_0' =$	−0·939 90	$-\phi_2' =$	−0·302 38
$\delta_0 = \beta_{0,0} =$	0·500 00	Check		$\delta_0 = \beta_{0,0} =$	0·504 14	$\delta_2 = \beta_{2,2} =$	0·160 16
$\log \beta_{0,0} =$	9·698 97			$\log \beta_{0,0} =$	9·702 55	$\log \beta_{2,2} =$	9·204 54
$\log (1 : \beta_{0,0}) =$	0·301 03	$\log (1 : \beta_{0,0}) =$	0·301 03	$\log (1 : \beta_{0,0}) =$	0·297 45	$\log (1 : \beta_{2,2}) =$	0·795 46
$\log r_0 =$	1·840 42	$\log h_0 =$	2·000 00	$\log r_0 =$	1·840 42	$\log r_2 =$	2·334 85
$\log \phi_0' =$	9·974 99			$\log \phi_0' =$	9·973 08	$\log \phi_2' =$	9·480 56
$\log (a_0' - r_0) =$	2·116 44	$\log a_0' =$	2·301 03	$\log (a_0' - r_0) =$	2·110 95	$\log (a_2' - r_2) =$	2·610 87
$a_0' - r_0 =$	130·750	$a_0' =$	200·000	$a_0' - r_0 =$	129·107	$a_2' - r_2 =$	408·205
$+r_0 - r_2 - d_1 =$	277·445	$-r_2 - d_1 =$	208·195	$+r_0 - r_2 - d_1 =$	277·445	$+r_2 - r_0 - d_1 =$	277·445
$a_0' - r_2 - d_1 =$	408·195	$a_0' - r_2 - d_1 =$	408·195	$a_0' - r_2 - d_1 =$	406·552	$a_2' - r_0 - d_1 =$	685·650
$\log (a_0' - r_2 - d_1) =$	2·610 87			$\log (a_0' - r_2 - d_1) =$	2·609 12	$\log (a_2' - r_0 - d_1) =$	2·836 10
$\log \beta_{0,0} =$	9·698 97			$\log \beta_{0,0} =$	9·702 55	$\log \beta_{2,2} =$	9·204 54
$\log (1 : r_2) =$	7·665 15*n*			$\log (1 : r_2) =$	7·665 15*n*	$\log (1 : r_0) =$	8·159 58*n*
$\log \phi_2 =$	9·974 99*n*			$\log \phi_2 =$	9·976 82*n*	$\log \phi_0 =$	0·200 22*n*
$\log (n_{+1} : n_{+3}) =$	0·184 59			$\log (n_{+1} : n_{+3}) =$	0·186 50	$\log (n_{+1} : n_{-1}) =$	0·184 59
$\log \phi_2' =$	0·159 58*n*			$\log \phi_2' =$	0·163 32*n*	$\log \phi_0' =$	0·384 81*n*

[*Continued over.*

Crown: $_{D}n_{+1} = 1·529\,64$ $\log {}_{D}n_{+1} = 0·184\,59$ $d_1 = 8;$ $r_0 = +69·250$ $\log r_0 = 1·840\,42$

$_{F}n_{+1} = 1·536\,39$ $\log {}_{F}n_{+1} = 0·186\,50$ $r_2 = -216·195$ $\log r_2 = 2·334\,85\,n$

Paraxial Rays.

D.				F.		Lens Reversed: D.	
$\phi_2 =$	$-0·944\,04$			$\phi_2 =$	$-0·948\,03$	$\phi_0 =$	$-1·585\,70$
$-\phi_2' =$	$1·444\,04$			$-\phi_2' =$	$1·456\,53$	$-\phi_0' =$	$2·425\,55$
$\delta_2 =$	$0·500\,00$			$\delta_2 =$	$0·508\,50$	$\delta_0 =$	$0·839\,85$
$\delta_0 =$	$0·500\,00$			$\delta_0 =$	$0·504\,14$	$\delta_2 =$	$0·160\,16$
$\beta_{0,2} =$	$1·000\,00$	Check		$\beta_{0,2} =$	$1·012\,64$	$\beta_{2,0} =$	$1·000\,01$
$\log \beta_{0,2} =$	$0·000\,00$	$a_0' - d_1 =$	$192·000$	$\log \beta_{0,2} =$	$0·005\,46$	$\log \beta_{2,0} =$	$0·000\,00$
$\log (1 : \beta_{0,2}) =$	$0·000\,00$	$\log (a_0' - d_1) =$	$2·283\,30$	$\log (1 : \beta_{0,2}) =$	$9·994\,54$	$\log (1 : \beta_{2,0}) =$	$0·000\,00$
$\log r_2 =$	$2·334\,85\,n$	$\log \beta_{0,0} =$	$9·698\,97$	$\log r_2 =$	$2·334\,85\,n$	$\log r_0 =$	$1·840\,42\,n$
$\log \phi_2' =$	$0·159\,58\,n$	$\log h_2 =$	$1·982\,27$	$\log \phi_2' =$	$0·163\,32\,n$	$\log \phi_0' =$	$0·384\,81\,n$
$\log (a_2' - r_2) =$	$2·494\,43$	$\log (1 : \beta_{0,2}) =$	$0·000\,00$	$\log (a_2' - r_2) =$	$2·492\,71$	$\log (a_0' - r_0) =$	$2·225\,23$
$a_2' - r_2 =$	$312·195$	$\log a_2' =$	$1·982\,27$	$a_2' - r_2 =$	$310·963$	$a_0' - r_0 =$	$167·969$
$r_2 =$	$-216·195$	$a_2' =$	$96·000$	$r_2 =$	$-216·195$	$r_0 =$	$-\ 69·250$
$a_2' =$	$96·000$			$a_2' =$	$94·768$	$a_0' =$	$98·719$
$\log h_0 =$	$2·000\,00$			$\log h_0 =$	$2·000\,00$	$\log h_2 =$	$2·000\,00$
$\log (1 : \beta_{0,2}) =$	$0·000\,00$			$\log (1 : \beta_{0,2}) =$	$9·994\,54$	$\log (1 : \beta_{2,0}) =$	$0·000\,00$
$\log f_{0,2} =$	$2·000\,00$			$\log f_{0,2} =$	$1·994\,54$	$\log f_{2,0} =$	$2·000\,00$
$f_{0,2} =$	$100·000$			$f_{0,2} =$	$98·751$	$f_{2,0} =$	$100·000$
$a_2' =$	$96·000$			$a_2' =$	$94·768$	$a_0' =$	$98·719$
$-f_{0,2} =$	$-100·000$			$-f_{0,2} =$	$-\ 98·751$	$-f_{2,0} =$	$-100·000$
$e_{0,2} =$	$-\ 4·000$			$e_{0,2} =$	$-\ 3·983$	$e_{2,0} =$	$-\ 1·281$

The ray, which is incident parallel to the axis at a height $h_0 = 15$, is taken as the rim ray, and its path is calculated for both the colours D and F; similarly, for the $\frac{2}{3}$-rim ray, where $h_0 = 10$, the calculations are made for the two colours D and F.

	Rim Ray: $h = 15$.		$\frac{2}{3}$-Rim Ray: $h = 10$.	
	D.	F.	D.	F.
$\log h_0$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : r_0)$	8·159 58	8·159 58	8·159 58	8·159 58
$\log \sin \phi_0$	9·335 67	9·335 67	9·159 58	9·159 58
$\log (n_{-1} : n_{+1})$	9·815 41	9·813 50	9·815 41	9·813 50
$\log \sin \phi_0'$	9·151 08	9·149 17	8·974 99	8·973 08
ϕ_0	12° 30′ 35″	12° 30′ 35″	8° 18′ 10·0″	8° 18′ 10·0″
$-\phi_0'$	− 8 8 27	− 8 6 17	−5 25 1·2	−5 23 35·4
$\delta_0 = \beta_{0,0}$	4 22 8	4 24 18	2 53 8·8	2 54 34·6
$\log \sin \beta_{0,0}$	8·881 83	8·885 40	8·701 96	8·705 53
$\log (1 : \sin \beta_{0,0})$	1·118 17	1·114 60	1·298 04	1·294 47
$\log \sin \phi_0'$	9·151 08	9·149 17	8·974 99	8·973 08
$\log r_0$	1·840 42	1·840 42	1·840 42	1·840 42
$\log (a_0' - r_0)$	2·109 67	2·104 19	2·113 45	2·107 97
$a_0' - r_0$	128·727	127·113	129·852	128·224
$+r_0 - r_2 - d_1$	277·447	277·447	277·447	277·447
$a_0' - r_2 - d_1$	406·174	404·560	407·299	405·671
$\log (a_0' - r_2 - d_1)$	2·608 71	2·606 98	2·609 91	2·608 17
$\log \sin \beta_{0,0}$	8·881 83	8·885 40	8·701 96	8·705 53
$\log (1 : r_2)$	7·665 15*n*	7·665 15*n*	7·665 15*n*	7·665 15*n*
$\log \sin \phi_2$	9·155 69*n*	9·157 53*n*	8·977 02*n*	8·978 85*n*
$\log (n_{+1} : n_{+3})$	0·184 59	0·186 50	0·184 59	0·186 50
$\log \sin \phi_2'$	9·340 28*n*	9·344 03*n*	9·161 61*n*	9·165 35*n*

[*Continued over.*

Computation for the two colours D and F (*Continued*)

	Rim Ray: $h = 15$.		⅔-Rim Ray: $h = 10$.	
	D.	F.	D.	F.
ϕ_2	− 8° 13′ 42″	− 8° 15′ 48″	−5° 26′ 32·9″	−5° 27′ 55·9″
$-\phi_2'$	12 38 44	12 45 25	8 20 31·1	8 24 52·7
δ_2	4 25 2	4 29 37	2 53 58·2	2 56 56·8
$\delta_0 = \beta_{0,0}$	4 22 8	4 24 18	2 53 8·8	2 54 34·6
$\beta_{0,2}$	8 47 10	8 53 55	5 47 7·0	5 51 31·4
$\log \sin \beta_{0,2}$	9·183 97	9·189 45	9·003 46	9·008 92
$\log (1 : \sin \beta_{0,2})$	0·816 03	0·810 55	0·996 54	0·991 08
$\log \sin \phi_2'$	9·340 28n	9·344 03n	9·161 61n	9·165 35n
$\log r_2$	2·334 85n	2·334 85n	2·334 85n	2·334 85n
$\log (a_2' - r_2)$	2·491 16	2·489 43	2·493 00	2·491 28
$a_2' - r_2$	309·856	308·624	311·172	309·942
r_2	−216·197	−216·197	−216·197	−216·197
a_2'	93·659	92·427	94·975	93·745
$\log h_0$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : \tan \beta_{0,2})$	0·810 90	0·805 29	0·994 32	0·988 80
$\log f_{0,2}$	1·986 99	1·981 38	1·994 32	1·988 80
$f_{0,2}$	97·048 8	95·803	98·701	97·454
a_2'	93·659	92·427	94·975	93·745
$-f_{0,2}$	− 97·049	− 95·803	− 98·701	− 97·454
$e_{0,2}$	− 3·390	− 3·376	− 3·726	− 3·709

II. Convergent lens of crown glass having the refractive indices $n_{-1} = n_{+3} = 1$, ${}_{D}n_{+1} = 1{\cdot}529\,64$, and ${}_{F}n_{+1} = 1{\cdot}536\,39$. Let the total deviation of the lens $\beta_{0,2} = 1$, the first radius $r_0 = +53{\cdot}0$, and the second radius r_2 be very large, $= 72748$. Let the thickness of the lens, $d_1 = 8$.

Crown: ${}_{D}n_{+1} = 1{\cdot}529\,64$ $\log {}_{D}n_{+1} = 0{\cdot}184\,59$ $d_1 = 8;$ $r_0 = +53{\cdot}0$ $\log r_0 = 1{\cdot}724\,28$

${}_{F}n_{+1} = 1{\cdot}536\,39$ $\log {}_{F}n_{+1} = 0{\cdot}186\,50$ $r_2 = -72\,748{\cdot}0$ $\log r_2 = 4{\cdot}861\,82\,n$

	Paraxial Ray: D.		Paraxial Ray: F.	
	$r_0 = 53{\cdot}0$ $r_2 = -72\,748{\cdot}0$	Lens Reversed: $r_0 = 72\,748{\cdot}0$ $r_2 = -53{\cdot}0$		
$\log h_0$	2·000 00	2·000 00	$\log h_0$	2·000 00
$\log(1 : r_0)$	8·275 72	5·138 18	$\log(1 : r_0)$	8·275 72
$\log \phi_0$	0·275 72	7·138 18	$\log \phi_0$	0·275 72
$\log(n_{-1} : n_{+1})$	9·815 41	9·815 41	$\log(n_{-1} : n_{+1})$	9·813 50
$\log \phi_0'$	0·091 13	6·953 59	$\log \phi_0'$	0·089 22
ϕ_0	1·886 8	0·001 374 6	ϕ_0	1·886 8
$-\phi_0'$	−1·233 5	−0·000 898 7	$-\phi_0'$	−1·228 1
$\delta_0 = \beta_{0,0}$	0·653 3	0·000 475 9	$\delta_0 = \beta_{0,0}$	0·658 7
$\log \beta_{0,0}$	9·815 11	6·677 52	$\log \beta_{0,0}$	9·818 69
$\log(1 : \beta_{0,0})$	0·184 89	3·322 48	$\log(1 : \beta_{0,0})$	0·181 31
$\log h_0$	2·000 00	2·000 00	$\log h_0$	2·000 00
$\log a_0'$	2·184 89	5·322 48	$\log a_0'$	2·181 31
a_0'	153·07	210 126·0	a_0'	151·8
$-r_2 - d_1$	72 740·00	45·0	$-r_2 - d_1$	72 740·0
$a_0' - r_2 - d_1$	72 893·07	210 171·0	$a_0' - r_2 - d_1$	72 891·8
$\log(a_0' - r_2 - d_1)$	4·862 69	5·322 57	$\log(a_0' - r_2 - d_1)$	4·862 68
$\log \beta_{0,0}$	9·815 11	6·677 52	$\log \beta_{0,0}$	9·818 69
$\log(1 : r_2)$	5·138 18 *n*	8·275 72 *n*	$\log(1 : r_2)$	5·138 18*n*
$\log \phi_2$	9·815 98 *n*	0·275 81 *n*	$\log \phi_2$	9·819 55*n*
$\log(n_{+1} : n_{+3})$	0·184 59	0·184 59	$\log(n_{+1} : n_{+3})$	0·186 50
$\log \phi_2'$	0·000 57 *n*	0·460 40 *n*	$\log \phi_2'$	0·006 05*n*

Check ($r_0 = 53{\cdot}0$ $r_2 = -72\,748{\cdot}0$)

Check	
$\log(1 : \beta_{0,0})$	0·184 89
$\log \phi_0'$	0·091 13
$\log r_0$	1·724 28
$\log(a_0' - r_0)$	2·000 30
$a_0' - r_0$	100·07
$r_0 - r_2 - d_1$	72 793·00
$a_0' - r_2 - d_1$	72 893·07
$\log(a_0' - r_2 - d_1)$	4·862 69

Check (Lens Reversed)

Check	
$\log(1 : \beta_{0,0})$	3·322 48
$\log \phi_0'$	6·953 59
$\log r_0$	4·861 82
$\log(a_0' - r_0)$	5·137 89
$a_0' - r_0$	137 369·5
$r_0 - r_2 - d_1$	72 793·0
$a_0' - r_2 - d_1$	210 162·5
$\log(a_0' - r_2 - d_1)$	5·322 56

[*Continued over.*

Crown: ${}_{D}n_{+1} = 1·529\,64$ $\log {}_{D}n_{+1} = 0·184\,59$ $d_1 = 8;$ $r_0 = +53·0$ $\log r_0 = 1·724\,28$

${}_{F}n_{+1} = 1·536\,39$ $\log {}_{F}n_{+1} = 0·186\,50$ $r_2 = -72\,748·0$ $\log r_2 = 4·861\,82\,n$

Paraxial Ray: D.								Paraxial Ray: F.	
$r_0 = 53·0$ $r_2 = -72\,748·0$				Lens Reversed: $r_0 = 72\,748·0$ $r_2 = -53·0$					
ϕ_2	−0·654 6			ϕ_2	−1·887 17			ϕ_2	−0·660 01
$-\phi_2'$	1·001 3			$-\phi_2'$	2·886 69			$-\phi_2'$	1·014 03
δ_2	0·346 7			δ_2	0·999 52			δ_2	0·354 02
δ_0	0·653 3			δ_0	0·000 48			δ_0	0·658 70
$\beta_{0,2}$	1·000 0			$\beta_{0,2}$	1·000 00			$\beta_{0,2}$	1·012 72
$a_0' - d_1$	145·07	Check		$a_0' - d_1$	210 118·0	Check		$a_0' - d_1$	143·813
$\log (a_0' - d_1)$	2·161 58	$\log (1 : \beta_{0,2})$	0·000 00	$\log (a_0' - d_1)$	5·322 46	$\log (1 : \beta_{0,2})$	0·000 00	$\log (a_0' - d_1)$	2·157 80
$\log \beta_{0,0}$	9·815 11	$\log \phi_2'$	0·000 57*n*	$\log \beta_{0,0}$	6·677 52	$\log \phi_2'$	0·460 40*n*	$\log \beta_{0,0}$	9·818 69
$\log h_2$	1·976 69	$\log r_2$	4·861 82*n*	$\log h_2$	1·999 98	$\log r_2$	1·724 28*n*	$\log h_2$	1·976 49
$\log (1 : \beta_{0,2})$	0·000 00	$\log (a_2' - r_2)$	4·862 39	$\log (1 : \beta_{0,2})$	0·000 00	$\log (a_2' - r_2)$	2·184 68	$\log (1 : \beta_{0,2})$	9·994 51
$\log a_2'$	1·976 69	$a_2' - r_2$	72 843·0	$\log a_2'$	1·999 98	$a_2' - r_2$	152·976	$\log a_2'$	1·971 00
a_2'	94·774	r_2	72 748·0	a_2'	99·995 4	r_2	53·000	a_2'	93·540 5
		a_2'	95·0			a_2'	99·996		
$\log h_0$	2·000 00			$\log h_0$	2·000 00			$\log h_0$	2·000 00
$\log (1 : \beta_{0,2})$	0·000 00			$\log (1 : \beta_{0,2})$	0·000 00			$\log (1 : \beta_{0,2})$	9·994 51
$\log f_{0,2}$	2·000 00			$\log f_{0,2}$	2·000 00			$\log f_{0,2}$	1·994 51
a_2'	94·774			a_2'	99·995 4			a_2'	93·540 5
$-f_{0,2}$	−100·000			$-f_{0,2}$	−100·000 0			$-f_{0,2}$	−98·744
$e_{0,2}$	− 5·226			$e_{0,2}$	− 0·004 6			$e_{0,2}$	− 5·203 5

Computation of the rim and $\frac{2}{3}$-rim rays for the colours D and F.

	Rim Ray: $h_0 = 15$.		$\frac{2}{3}$-Rim Ray: $h_0 = 10$.	
	D.	F.	D.	F.
$\log h_0$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : r_0)$	8·275 72	8·275 72	8·275 72	8·275 72
$\log \sin \phi_0$	9·451 81	9·451 81	9·275 72	9·275 72
$\log (n_{-1} : n_{+1})$	9·815 41	9·813 50	9·815 41	9·813 50
$\log \sin \phi_0'$	9·267 22	9·265 31	9·091 13	9·089 22
$\phi_0 = \eta_0$	16° 26′ 25″	16° 26′ 25″	10° 52′ 32″	10° 52′ 32″
$-\phi_0'$	−10 39 45	−10 36 54	− 7 5 7	− 7 3 15
$\delta_0 = \beta_{0,0}$	5 46 40	5 49 31	3 47 25	3 49 17
$\log \tan \beta_{0,0}$	9·005 11	9·008 69	8·821 18	8·824 74
$\log (1 : \tan \beta_{0,0})$	0·994 89	0·991 31	1·178 82	1·175 26
$\log h_0$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (h_0 : \tan \beta_{0,0})$	2·170 98	2·167 40	2·178 82	2·175 26
$\log \sin \frac{\eta_0}{2}$	9·155 27		8·976 75	
$\log \sin^2 \frac{\eta_0}{2}$	8·310 54		7·953 50	
$\log 2$	0·301 03		0·301 03	
$\log r_0$	1·724 28		1·724 28	
$\log \left(2r_0 \sin^2 \frac{\eta_0}{2}\right)$	0·335 85		9·978 81	
$h_0 : \tan \beta_{0,0}$	148·245	147·028	150·945	149·714
$2r_0 \sin^2 \frac{\eta_0}{2}$	2·167	2·167	0·952	0·952
a_0'	150·412	149·195	151·897	150·666
$-r_2 - d_1$	72 740·0	72 740·0	72 740·0	72 740·0
$a_0' - r_2 - d_1$	72 890·4	72 889·2	72 891·897	72 890·666

[*Continued over.*

Computation for the colours D and F (*Continued*)

	Rim Ray: $h_0 = 15$.		$\frac{2}{3}$-Rim Ray: $h_0 = 10$.	
	D.	F.	D.	F.
$\log (a_0' - r_2 - d_1)$	4·862 67	4·862 66	4·862 68	4·862 67
$\log \sin \beta_{0,0}$	9·002 90	9·006 45	8·820 23	8·823 78
$\log (1 : r_2)$	5·138 18*n*	5·138 18*n*	5·138 18*n*	5·138 18*n*
$\log \sin \phi_2$	9·003 75*n*	9·007 29*n*	8·821 09*n*	8·824 63*n*
$\log (n_{+1} : n_{+3})$	0·184 59	0·186 50	0·184 59	0·186 50
$\log \sin \phi_2'$	9·188 34*n*	9·193 79*n*	9·005 68*n*	9·011 13*n*
ϕ_2	−5° 47′ 21″	−5° 50′ 12″	−3° 47′ 52″	−3° 49′ 44″
$-\phi_2'$	8 52 32	8 59 19	5 48 54	5 53 19
δ_2	3 5 11	3 9 7	2 1 2	2 3 35
δ_0	5 46 40	5 49 31	3 47 25	3 49 17
$\beta_{0,2}$	8 51 51	8 58 38	5 48 27	5 52 52
ϕ_2	−5 47 21	−5 50 12	−3 47 52	−3 49 44
$\beta_{0,0}$	5 46 40	5 49 31	3 47 25	3 49 17
η_2	−0 0 41	−0 0 41	−0 0 27	−0 0 27
$\log \sin \frac{\eta_2}{2}$	5·996 89*n*		5·820 61*n*	
$\log \sin^2 \frac{\eta_2}{2}$	1·993 78		1·641 22	
$\log (-r_2)$	4·861 82		4·861 82	
$\log 2$	0·301 03		0·301 03	
$\log \left(-2r_2 \sin^2 \frac{\eta_2}{2}\right)$	7·156 63		6·804 07	
$-2r_2 \sin^2 \frac{\eta_2}{2}$	0·001 4	0·001 4	0·000 64	0·000 6
$a_0' - d_1$	142·412	141·195	143·897 4	142·666 4
$a_0' - d_1 - 2r_2 \sin^2 \frac{\eta_2}{2}$	142·413 4	141·196 4	143·898	142·667

Computation for the colours D and F (*Continued*)

	Rim Ray: $h_0 = 15$.		⅔-Rim Ray: $h_0 = 10$.	
	D.	F.	D.	F.
$\log\left(a_0' - d_1 - 2r_2 \sin^2 \frac{\eta_2}{2}\right)$	2·153 55	2·149 82	2·158 05	2·154 32
$\log \tan \beta_{0,0}$	9·005 11	9·008 69	8·821 18	8·824 74
$\log h_2$	1·158 66	1·158 51	0·979 23	0·979 06
$\log (1 : \tan \beta_{0,2})$	0·807 00	0·801 54	0·992 64	0·987 13
$\log (h_2 : \tan \beta_{0,2})$	1·965 66	1·960 05	1·971 87	1·966 19
$h_2 : \tan \beta_{0,2}$	92·397 5	91·211 5	93·728 1	92·510 3
$2r_2 \sin^2 \frac{\eta_2}{2}$	− 0·001 4	− 0·001 4	− 0·000 6	− 0·000 6
a_2'	92·396 1	92·210 1	93·727 5	92·509 7
$\log h_0$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : \tan \beta_{0,2})$	0·807 00	0·801 54	0·992 64	0·987 13
$\log f_{0,2}$	1·983 09	1·977 63	1·992 64	1·987 13
a_2'	92·396 1	91·210 1	93·727 5	92·509 7
$-f_{0,2}$	−96·181 0	−94·979 5	−98·320 0	−97·080 0
$e_{0,2}$	− 3·784 9	− 3·769 4	− 4·592 5	− 4·570 3

III. Convergent lens of crown glass having the refractive indices $n_{-1} = n_{+3} = 1$, ${}_{D}n_{+1} = 1·529\,64$, and ${}_{F}n_{+1} = 1·536\,39$. Let the total deviation of the lens $\beta_{0,2} = 1$, and the first radius $= \infty$. Then the second radius r_2 becomes $= -52·966\,3$ for the thickness $d_1 = 8$.

Crown: ${}_{D}n_{+1} = 1·529\,64$ $\log {}_{D}n_{+1} = 0·184\,59$ $d_1 = 8$; $r_0 = \infty$
${}_{F}n_{+1} = 1·536\,39$ $\log {}_{F}n_{+1} = 0·186\,50$ $r_2 = -52·966\,3$ $\log r_2 = 1·724\,00n$

	Paraxial Ray.			Rim Ray: $h_0 = 15$.		$\frac{2}{3}$-Rim Ray: $h = 10$.	
	D.	F.		D.	F.	D.	F.
$\log h_2$	2·000 00	2·000 00	$\log h_2$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : r_2)$	8·276 00*n*	8·276 00*n*	$\log (1 : r_2)$	8·276 00*n*	8·276 00*n*	8·276 00*n*	8·276 00*n*
$\log \phi_2$	0·276 00*n*	0·276 00*n*	$\log \sin \phi_2$	9·452 09*n*	9·452 09*n*	9·276 00*n*	9·276 00*n*
$\log (n_{+1} : n_{+3})$	0·184 59	0·186 50	$\log (n_{+1} : n_{+3})$	0·184 59	0·186 50	0·184 59	0·186 50
$\log \phi_2'$	0·460 59*n*	0·462 50*n*	$\log \sin \phi_2'$	9·636 68*n*	9·638 59*n*	9·460 59*n*	9·462 50*n*
ϕ_2	−1·887 99	−1·887 99	ϕ_2	−16° 27′ 4″	−16° 27′ 4″	−10° 52′ 57·8″	−10° 52′ 57·8″
$-\phi_2'$	2·887 95	2·900 68	$-\phi_2'$	25 40 13	25 47 30	16 47 9·0	16 51 43·3
$\delta_2 = \beta_{0,2}$	0·999 96	1·012 69	$\delta_2 = \beta_{0,2}$	9 13 9	9 20 26	5 54 11·2	5 58 45·5
$\log \beta_{0,2}$	9·999 98	0·005 48	$\log \sin \beta_{0,2}$	9·204 69	9·210 32	9·012 19	9·017 74
$\log (1 : \beta_{0,2})$	0·000 02	9·994 52	$\log (1 : \sin \beta_{0,2})$	0·795 31	0·789 68	0·987 81	0·982 26
$\log \phi_2'$	0·460 59*n*	0·462 50*n*	$\log \sin \phi_2'$	9·636 68*n*	9·638 59*n*	9·460 59*n*	9·462 50*n*
$\log r_2$	1·724 00*n*	1·724 00*n*	$\log r_2$	1·724 00*n*	1·724 00*n*	1·724 00*n*	1·724 00*n*
$\log (a_2' - r_2)$	2·184 61	2·181 02	$\log (a_2' - r_2)$	2·155 99	2·152 27	2·172 40	2·168 76
$a_2' - r_2$	152·971	151·712	$a_2' - r_2$	143·215	141·994	148·730	147·489
r_2	− 52·966	− 52·966	r_2	− 52·966	− 52·966	− 52·966	− 52·966
a_2'	100·005	98·746	a_2'	90·249	89·028	95·764	94·523
$\log h_2$	2·000 00	2·000 00	$\log h_2$	1·176 09	1·176 09	1·000 00	1·000 00
$\log (1 : \beta_{0,2})$	0·000 02	9·994 52	$\log (1 : \tan \beta_{0,2})$	0·789 66	0·783 88	0·985 50	0·979 89
$\log f_{0,2}$	2·000 02	1·994 52	$\log f_{0,2}$	1·965 75	1·959 97	1·985 50	1·979 89
a_2'	100·005	98·746	a_2'	90·249	89·028	95·764	94·523
$-f_{0,2}$	−100·005	− 98·746	$-f_{0,2}$	− 92·417	− 91·195	− 96·716	− 95·475
$e_{0,2}$	0·000	0·000	$e_{0,2}$	− 2·168	− 2·167	− 0·952	− 0·952

IV. The same lens as above, but in the reversed position, so that $r_0 = +52{\cdot}966\,3$ and $r_2 = \infty$.

Crown: ${}_{D}n_{+1} = 1{\cdot}529\,64$ $\log {}_{D}n_{+1} = 0{\cdot}184\,59$ $d_1 = 8$; $r_0 = +52{\cdot}966\,3$ $\log r_0 = 1{\cdot}724\,00$

${}_{F}n_{+1} = 1{\cdot}536\,39$ $\log {}_{F}n_{+1} = 0{\cdot}186\,50$ $r_2 = \infty$

	Paraxial Ray.			Rim Ray: $h_0 = 15$.		$\frac{2}{3}$-Rim Ray: $h_0 = 10$.	
	D.	F.		D.	F.	D.	F.
$\log h_0$ $\log (1 : r_0)$	2·000 00 8·276 00	2·000 00 8·276 00	$\log h_0$ $\log (1 : r_0)$	1·176 09 8·276 00	1·176 09 8·276 00	1·000 00 8·276 00	1·000 00 8·276 00
$\log \phi_0$ $\log (n_{-1} : n_{+1})$	0·276 00 9·815 41	0·276 00 9·813 50	$\log \sin \phi_0$ $\log (n_{-1} : n_{+1})$	9·452 09 9·815 41	9·452 09 9·813 50	9·276 00 9·815 41	9·276 00 9·813 50
$\log \phi_0'$	0·091 41	0·089 50	$\log \sin \phi_0'$	9·267 50	9·265 59	9·091 41	9·089 50
ϕ_0 $-\phi_0'$	1·887 99 −1·234 27	1·887 99 −1·228 85	ϕ_0 $-\phi_0'$	16° 27′ 4″ −10 40 9	16° 27′ 4″ −10 37 19	10° 52′ 57·8″ − 7 5 23·7	10° 52′ 57·8″ − 7 3 31·1
$\delta_0 = \beta_{0,0}$	0·653 72	0·659 14	$\delta_0 = \beta_{0,0}$	5 46 55	5 49 45	3 47 34·1	3 49 26·7
$\log \beta_{0,0}$ $\log (1 : \beta_{0,0})$ $\log \phi_0'$ $\log r_0$	9·815 39 0·184 61 0·091 41 1·724 00	9·818 98 0·181 02 0·089 50 1·724 00	$\log \sin \beta_{0,0}$ $\log (1 : \sin \beta_{0,0})$ $\log \sin \phi_0'$ $\log r_0$	9·003 21 0·996 79 9·267 50 1·724 00	9·006 73 0·993 27 9·265 59 1·724 00	8·820 52 1·179 48 9·091 41 1·724 00	8·824 08 1·175 92 9·089 50 1·724 00
$\log (a_0' - r_0)$	2·000 02	1·994 52	$\log (a_0' - r_0)$	1·988 29	1·982 86	1·994 89	1·989 42
$a_0' - r_0$ $r_0 - d_1$	100·005 44·966	98·746 1 44·966 3	$a_0' - r_0$ $r_0 - d_1$	97·339 7 44·966 3	96·130 2 44·966 3	98·830 3 44·966 3	97·593 3 44·966 3
$a_0' - d_1$	144·971	143·712	$a_0' - d_1$	142·306 0	141·096 5	143·797	142·560

[*Continued on p. 94.*

Computation of the lens in the reversed position (*Continued*)

Crown: ${}_{\mathrm{D}}n_{+1} = 1{\cdot}529\,64$ $\log {}_{\mathrm{D}}n_{+1} = 0{\cdot}184\,59$ $d_1 = 8;$ $r_0 = +52{\cdot}966\,3$ $\log r_0 = 1{\cdot}724\,00$

${}_{\mathrm{F}}n_{+1} = 1{\cdot}536\,39$ $\log {}_{\mathrm{F}}n_{+1} = 0{\cdot}186\,50$ $r_2 = \infty$

	Paraxial Ray.			Rim Ray: $h_0 = 15$.		$\frac{2}{3}$-Rim Ray: $h_0 = 10$.	
	D.	F.		D.	F.	D.	F.
$\log \phi_2 = \log(-\beta_{0,0})$ $\log(n_{+1} : n_{+3})$	$9{\cdot}815\,39n$ $0{\cdot}184\,59$	$9{\cdot}818\,98n$ $0{\cdot}186\,50$	$\log(-\sin\beta_{0,0})$ $\log(n_{+1} : n_{+3})$	$9{\cdot}003\,21n$ $0{\cdot}184\,59$	$9{\cdot}006\,73n$ $0{\cdot}186\,50$	$8{\cdot}820\,52n$ $0{\cdot}184\,59$	$8{\cdot}824\,08n$ $0{\cdot}186\,50$
$\log \phi_2'$	$9{\cdot}999\,98n$	$0{\cdot}005\,48n$	$\log \sin \phi_2'$	$9{\cdot}187\,80n$	$9{\cdot}193\,23n$	$9{\cdot}005\,11n$	$9{\cdot}010\,58n$
$-\phi_2' = \beta_{0,2}$	$0{\cdot}999\,954$	$1{\cdot}012\,70$	$-\phi_2' = \beta_{0,2}$	$8^\circ\ 51'\ 52''$	$8^\circ\ 58'\ 37''$	$5^\circ\ 48'\ 26{\cdot}4''$	$5^\circ\ 52'\ 52{\cdot}3''$
$\log(a_0' - d_1)$ $\log \beta_{0,0}$ $\log(1 : \beta_{0,2})$	$2{\cdot}161\,28$ $9{\cdot}815\,39$ $0{\cdot}000\,02$	$2{\cdot}157\,49$ $9{\cdot}818\,98$ $9{\cdot}994\,52$	$\log(a_0' - d_1)$ $\log \tan \beta_{0,0}$ $\log(1 : \tan \beta_{0,2})$	$2{\cdot}153\,22$ $9{\cdot}005\,43$ $0{\cdot}806\,99$	$2{\cdot}149\,52$ $9{\cdot}008\,99$ $0{\cdot}801\,42$	$2{\cdot}157\,75$ $8{\cdot}821\,47$ $0{\cdot}992\,66$	$2{\cdot}154\,00$ $8{\cdot}825\,05$ $0{\cdot}987\,13$
$\log a_2'$	$1{\cdot}976\,69$	$1{\cdot}970\,99$	$\log a_2'$	$1{\cdot}965\,64$	$1{\cdot}959\,92$	$1{\cdot}971\,88$	$1{\cdot}966\,18$
$\log h_0$ $\log(1 : \beta_{0,2})$	$2{\cdot}000\,00$ $0{\cdot}000\,02$	$2{\cdot}000\,00$ $9{\cdot}994\,52$	$\log h_0$ $\log(1 : \tan \beta_{0,2})$	$1{\cdot}176\,09$ $9{\cdot}193\,01$	$1{\cdot}176\,09$ $9{\cdot}198\,58$	$1{\cdot}000\,00$ $0{\cdot}992\,66$	$1{\cdot}000\,00$ $0{\cdot}987\,13$
$\log f_{0,2}$	$2{\cdot}000\,02$	$1{\cdot}994\,52$	$\log f_{0,2}$	$1{\cdot}983\,08$	$1{\cdot}977\,51$	$1{\cdot}992\,66$	$1{\cdot}987\,13$
a_2' $-f_{0,2}$	$94{\cdot}774\,2$ $-100{\cdot}005$	$93{\cdot}538\,4$ $-98{\cdot}746\,1$	a_2' $-f_{0,2}$	$92{\cdot}393\,2$ $-96{\cdot}178\,9$	$91{\cdot}186\,4$ $-94{\cdot}953\,3$	$93{\cdot}730\,3$ $-98{\cdot}324\,1$	$92{\cdot}508\,1$ $-97{\cdot}080\,0$
$e_{0,2}$	$-\ 5{\cdot}230\,8$	$-\ 5{\cdot}207\,7$	$e_{0,2}$	$-\ 3{\cdot}785\,7$	$-\ 3{\cdot}766\,9$	$-\ 4{\cdot}593\,8$	$-\ 4{\cdot}571\,9$

Discussion of the Focal-length Aberrations

In the manner indicated in the foregoing examples the paths of various rays have been calculated for a series of lenses all having the same true focal length, and the results so obtained will be used to demonstrate the effect of alterations of the various chosen dimensions on the aberrations of the image. Thus, for example, the effective aperture, the thickness of the lens, the form of the lens, and, finally, the type of glass may be altered in various ways. As a result of these variations, while the true focal length is kept unaltered, widely different aberrations may be produced in the image points formed by the lenses.

(*a*) *The Influence of the Effective Lens Aperture* (*Incidence Height*) *upon the Focal-length Aberrations*

In Table I (p. 101), there are given the results of computation for a lens in which the paraxial ray suffers equal deviations of 0·5 at the two surfaces. In addition there are indicated for the same lens the results obtained from the computation of rays of two colours, those in the first column being for D rays, and those in the third for F rays. In the second column the differences between the two colours are indicated.

The lens has its true focal length (namely, 100 mm.) for the D rays, and the computations are made for rays of this colour incident at eight different heights. Commencing with the highest, the incidence heights chosen are 22·5 mm., 15·0 mm., 10·0 mm., 6·6667 mm., 4·4444 mm., 2·9630 mm., 1·9753 mm., and for the last paraxial ray 0·000485 mm. The last incidence height corresponds with a semi angular aperture of one second. It will be noted that in the case of the first seven rays each incidence height is two-thirds of the succeeding larger incidence height. The reason for this choice is that it facilitates the determination of the smallest circle of confusion. The grounds for this statement will be dealt with later.

The table comprises two sections. The first gives the particulars, and the second the results, of the computation for the lens. Lines 1 to 8 in the first section of Group A are the chosen incidence heights. The corresponding focal lengths—that is, the distances of the focal points from the principal points—are indicated in the second section of the group. The first line in the D column of this second section represents *the true focal length* of the lens. Lines 9 to 16 in the second section of Group B represent the *back focal lengths*—that is, the distance of the point of intersection of the emergent ray with the axis, measured from the vertex of the last lens surface.

Under Group C there are indicated the *abscissæ of the principal or virtual emergence points*, reckoned from the vertex of the last surface. Lines 17 to 24 of the first subdivision of this group represent the ideal values, and lines 25 to 32 of the second subdivision the actual values. These ideal values are calculated, as indicated on p. 39, on the assumption that the principal or virtual emergence points lie upon a sphere drawn through the axial principal point, with the focus as centre and the true focal length as radius.

The ideal virtual emergence points corresponding with the specified incidence heights when projected on the axis lie behind the principal point of the axial ray at the following distances:—

$$+2{\cdot}439 \text{ mm.}, +1{\cdot}106 \text{ mm.}, +0{\cdot}496 \text{ mm.}, +0{\cdot}241 \text{ mm.}, +0{\cdot}099 \text{ mm.}, +0{\cdot}044 \text{ mm.}, +0{\cdot}020 \text{ mm.}$$

The first corresponds with the greatest incidence height, and in all the direction of the incidence light is regarded as positive. These distances are inserted in the first section of particulars opposite their corresponding incidence heights, and in the second section there are indicated, with their appropriate signs, the distances of the virtual emergence points from the vertex.

Since the true focal length (= 100 mm.) is chosen for the D line, the ideal values need only be obtained for this colour, and the differences in the positions of the virtual emergence points of the focal lengths for the two colours are to be regarded as errors.

The actual distances from the vertex of the projections of the virtual emergence points are obtained from the trigonometrical computation by means of the formula

$$(88) \qquad e = a_2' - f_{0,2}.$$

Group D of the tables gives the semi angular aperture, that is, the angle at which the emergent peripheral rays intersect the axis. Lines 33 to 40 represent the ideal, and lines 41 to 48 the actual values. The ideal values represent the angle through which a ray incident at a particular height must be deviated towards the axis in order that the distance of the virtual emergence point from the point of intersection of the ray with the axis may be equal to the true focal length of the lens, the point of intersection with the axis being the same for all the rays. This ideal angle is obtained from the formula

$$(89) \qquad \sin \beta_{0,2} = \frac{h_0}{f_{0,2}}.$$

Commencing with the largest incidence height, the corresponding angular values are thus obtained, namely:

46810·3 sec. 31056·9 sec. 20661·0 sec. 13761·2 sec.
9170·3 ,, 6112·4 ,, 4074·6 ,, 1·0 ,,

In the subsequent groups, E and F, the errors in position of the focal points and the virtual emergent points are tabulated. These errors are due to two causes:

1. The unequal refrangibilities of the rays incident at the *same height.*
2. The unproportional deviation, due to the incidence of rays at different heights.

As already indicated in Chapter III, the first-mentioned errors are termed *chromatic aberrations* when they concern the focal points, and *chromatic differences of magnification* when they concern the virtual emergence points. The errors indicated under E (2) are called *spherical aberrations* when they concern a variation in the position of the focus, and *distortion* when they concern a variation of the virtual emergence-point positions.

The chromatic and spherical aberrations have been expressed in two ways, namely, as linear and as angular differences. From a closer examination of the table it will be seen that the latter data provide a more correct idea of the influence of the errors upon the definition of the image, that is, the diameter of the image point. This follows from the fact that the sizes of objects are appreciated by the eye in an angular sense, while longitudinal differences provide no measure of the definition or diameter of the image point.

Compare, for example, the chromatic aberrations tabulated in groups E (1) and F (1). The angular error indicated under E (1) is a measure of the effect at the eye. The greater the angular difference the greater does the error appear to the eye, whereas, as indicated under F (1), the longitudinal difference of the chromatic aberration becomes smaller as the aperture increases, and an entirely misleading idea of the chromatic aberrations is thus produced.

The linear aberrations become smaller, for the reason that a definite fraction of the focal length continuously decreases in value, owing to the decrease in the focal length as the relative aperture increases.

Both the angular and linear values of the chromatic aberration may be obtained from approximate formulæ. Thus, for the angular aberration,

$$\triangle \beta_{0,2} = -\frac{\triangle n_1}{n_1 - 1} \cdot \frac{h_0}{f_{0,2}}, \tag{90}$$

and, for the linear aberration,

$$\triangle a_2' = +\frac{\triangle n_1}{n_1 - 1} \cdot f_{0,2}. \tag{91}$$

In the example already chosen the following values would be obtained:—

1. Error of $\beta_{0,2}$, i.e. $\triangle \beta_{0,2}$.

For h_0 = 0·000485	......	−0·0127″
1·9753	......	−51·92″
2·9630	......	−77·89″
4·4444	......	−116·83″
6·6667	......	−175·25″
10·0	......	−262·88″
15·0	......	−394·31″
22·5	......	−591·47″

2. Error of a_2', i.e. $\triangle a_2'$

+ 1·275 mm.

In order to determine the diameters of the image points formed by rays of different colours, as will be indicated later, it is necessary to use both the calculated angular and linear values of the aberrations.

Under groups E (2) and F (2) there are tabulated the spherical aberrations. In this case also the angular aberrations in group E (2) provide a truer idea of the visual effect than do the linear aberrations. Both the angular and linear spherical aberrations increase rapidly as the aperture of the lens increases, and the rate of increase is much greater than in the case of the chromatic aberrations.

The angular and linear values of the spherical aberration also can be obtained from approximate formulæ. Thus, for the angular aberration,

(92)

$$\triangle \beta_{0,2} = -\frac{n h_0^3}{2 f_{0,2}}\left[n\left(\frac{1}{r_0}-\frac{1}{r_2}\right)^2 - \frac{1+2n}{n r_0}\left(\frac{1}{r_0}-\frac{1}{r_2}\right) + \frac{2+n}{n^2 r_0^2}\right],$$

and for the linear aberration,

(93)

$$\triangle a_2' = \frac{n h_0^2 f_{0,2}}{2}\left[n\left(\frac{1}{r_0}-\frac{1}{r_2}\right)^2 - \frac{1+2n}{n r_0}\left(\frac{1}{r_0}-\frac{1}{r_2}\right) + \frac{2+n}{n^2 r_0^2}\right].$$

Using these approximate formulæ, as in the previous examples, the following values of the errors are obtained:—

1. Angular aberrations in seconds:

Incidence Heights.	$\delta_0 = \delta_2 = 0{\cdot}5$.
0·000485	0·000″
1·9753	−1·711″
2·9630	−5·775″
4·4444	−19·489″
6·6667	−65·78″
10·0	−221·99″
15·0	−749·22″
22·5	−2528·60″

2. Longitudinal aberrations:

Incidence Heights.	$\delta_0 = \delta_2 = 0{\cdot}5$.
0·000485	0·000 mm.
1·9753	0·042 ,,
2·9630	0·094 ,,
4·4444	0·213 ,,
6·6667	0·478 ,,
10·0	1·076 ,,
15·0	2·422 ,,
22·5	5·448 ,,

The linear errors at the principal points corresponding with different incidence heights are tabulated in Group F (2), the differences in the positions of the principal points for rays of the two colours—that is, the chromatic differences of magnification—being indicated in Section (*a*). This error for the lens under consideration is quite unimportant; its value is uncertain in the third decimal place.

The differences between the positions of the principal points of rim rays and that of the principal point of a D axial ray, that is, the values of the distortion, are tabulated in Section (*b*). For the two colours in question the distortion is almost the same, and it increases as the lens aperture becomes greater.

From a brief analysis of the results obtainable from the tables, the following rules may be enunciated:—

1. The absolute values of all aberrations increase in general with an increase in the effective aperture of the lens.
2. Aberrations at the focal point are in general greater than those at the principal point.
3. The angular values of the chromatic and spherical aberrations afford a truer idea of the visual effect produced than do the linear values.
4. As the lens aperture increases, the spherical aberrations increase more rapidly than the chromatic aberrations.
5. The angular values of the spherical aberration increase approximately as the third power of the incidence

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height, and those of the chromatic aberration approximately as the first power. The longitudinal values of the spherical aberration, on the other hand, increase approximately as the square of the incidence height, while those of the chromatic aberration are independent of the incidence height.

I. INFLUENCE OF THE EFFECTIVE LENS APERTURE UPON THE FOCAL-LENGTH ABERRATIONS

Particulars.		$d_1 = 8;\ r_0 = +69{\cdot}250;\ r_2 = -216{\cdot}195;\ \delta_0 = \delta_2 = +0{\cdot}5$		
		D.	Differences.	F.
		mm.		mm.
A. Focal lengths: $f_{0,2}$ for:	0·000 485	100·000		98·751
	1·975 3	99·949		98·696
	2·963 0	99·885		98·637
	4·444 4	99·745		98·496
	6·666 7	99·424		98·177
	10·000	98·701		97·454
	15·000	97·049		95·803
	22·500	93·195		91·960
B. Back focal lengths: a_2' for:	0·000 485	96·001		94·767
	1·975 3	95·958		94·724
	2·963 0	95·915		94·681
	4·444 4	95·800		94·567
	6·666 7	95·548		94·316
	10·000	94·975		93·745
	15·000	93·659		92·427
	22·500	90·550		89·323
C. Abscissæ of principal points:				
1. Ideal values:				
$e' = {}_{\text{Ax.}}(a_2' - f_{0,2}) =$				
$e^* = e' + 0{\cdot}000$ for	0·000 485	−3·999		
$e' + 0{\cdot}020$	1·975 3	−3·979		
$e' + 0{\cdot}044$	2·963 0	−3·955		
$e' + 0{\cdot}099$	4·444 4	−3·900		
$e' + 0{\cdot}241$	6·666 7	−3·758		
$e' + 0{\cdot}496$	10·000	−3·503		
$e' + 1{\cdot}106$	15·000	−2·893		
$e' + 2{\cdot}439$	22·500	−1·560		

[*Continued over.*

Particulars.		$d_1 = 8$; $r_0 = +69{\cdot}250$ $r_2 = -216{\cdot}195$ $\delta_0 = \delta_2 = +0{\cdot}5$		
		D.	Differences.	F.
		mm.		mm.
2. Actual values:				
$e = a_2' - f_{0,2}$ for:	0·000 485	−3·999		−3·984
	1·975 3	−3·991		−3·972
	2·963 0	−3·970		−3·956
	4·444 4	−3·945		−3·929
	6·666 7	−3·876		−3·861
	10·000	−3·726		−3·709
	15·000	−3·390		−3·376
	22·500	−2·645		−2·637
D. Semi angular aperture:				
1. Ideal values: $\beta'_{0,2}$ for:	0·000 485	1·000″		
	1·975 3	4 074·600		
	2·963 0	6 112·400		
	4·444 4	9 170·300		
	6·666 7	13 761·200		
	10·000	20 661·000		
	15·000	31 056·900		
	22·500	46 810·300		
2. Actual values: $\beta_{0,2}$ for:	0·000 485	1·000″		1·013″
	1·975 3	4 075·900		4 127·600
	2·963 0	6 116·790		6 194·170
	4·444 4	9 184·800		9 301·100
	6·666 7	13 810·000		13 985·000
	10·000	20 827·000		21 091·400
	15·000	31 630·000		32 035·000
	22·500	48 863·000		49 495·000
E. Angular errors:				
1. Chromatic aberration:				
${}_D\beta_{0,2} - {}_F\beta_{0,2}$ for:	0·000 485		− 0·013″	
	1·975 3		− 51·700	
	2·963 0		− 77·380	
	4·444 4		−116·300	
	6·666 7		−175·000	
	10·000		−264·400	
	15·000		−405·000	
	22·500		−632·000	
2. Spherical aberration:				
$\beta'_{0,2} - \beta_{0,2}$ for:	0·000 485	0·000″		− 0·013″
	1·975 3	− 1·300		− 53·000
	2·963 0	− 4·390		− 81·770
	4·444 4	− 14·500		− 130·800
	6·666 7	− 48·800		− 223·800
	10·000	− 166·000		− 430·400
	15·000	− 573·100		− 978·100
	22·500	−2 052·700		− 2684·700

Particulars.		$d_1 = 8$; $r_0 = +69·250$ $r_2 = -216·195$ $\delta_0 = \delta_2 = +0·5$		
		D.	Differences.	F.
F. Errors of the focal length:				
1. At the focal point:				
(*a*) Chromatic aberration:			mm.	
$_{D}a_2' - {}_{F}a_2'$ for:	0·000 485		+1·234	
	1·975 3		+1·234	
	2·963 0		+1·234	
	4·444 4		+1·233	
	6·666 7		+1·232	
	10·000		+1·230	
	15·000		+1·232	
	22·500		+1·227	
(*b*) Spherical aberration		mm.		mm.
$_{\text{Ax.}}a_2' - {}_{\text{Rim}}a_2'$ for:	0·000 485	0·000		0·000
	1·975 3	+0·043		+0·043
	2·963 0	+0·086		+0·086
	4·444 4	+0·201		+0·200
	6·666 7	+0·453		+0·451
	10·000	+1·026		+1·022
	15·000	+2·342		+2·340
	22·500	+5·450		+5·444
2. At the principal point:				
(*a*) Chromatic difference of			mm.	
magnification $_{D}e - {}_{F}e$ for:	0·000 485		−0·015	
	1·975 3		−0·019	
	2·963 0		−0·014	
	4·444 4		−0·016	
	6·666 7		−0·015	
	10·000		−0·017	
	15·000		−0·014	
	22·500		−0·008	
		mm.		mm.
(*b*) Distortion: $e^* - e$ for:	0·000 485	0·000		−0·015
	1·975 3	+0·012		−0·007
	2·963 0	+0·015		+0·001
	4·444 4	+0·045		+0·029
	6·666 7	+0·118		+0·103
	10·000	+0·223		+0·206
	15·000	+0·497		+0·483
	22·500	+1·085		+1·077

(*b*) *The Influence of the Lens Thickness on the Errors of the Focal Length*

In a manner similar to Table I, the results of computation for five lenses are summarized in Table II (p. 106).

For all five lenses, the true focal length of which is 100 mm., it is assumed that the deviations at the two lens faces are equal, that is, $\delta_0 = \delta_2 = 0{\cdot}5$. Only the separation of the surfaces, that is to say, the lens thickness, has been varied.

In the case of the first lens, the thickness chosen is very small, namely, 0·001 mm., while for the other lenses the thicknesses chosen are of increasing amount, namely, 8 mm., 48 mm., and 88 mm. For the last lens the thickness chosen is a maximum, namely, 138·5 mm., the lens being then a sphere. In the case of the last four lenses, the calculation has been made for incidence heights of 0·000485 mm., 10 mm., and 15 mm., but for the lens of 0·001 mm. thickness an incidence height of 0·000485 mm. has alone been taken, since rays incident at heights of 10 mm. and 15 mm. would not meet this last lens, as its aperture is only 0.325 mm.

For each of the five lenses the table is arranged in two columns, one for D and the other for F rays, and for each lens there are six groups lettered A to F. In the three first lines of Group A there are tabulated the focal lengths, while lines 4–6 of Group B indicate the back focal lengths. Lines 7–9 of Group C represent the ideal, and lines 10–12 the actual abscissæ of the principal points measured along the axis from the last refracting surface. Finally, in lines 13–15 of Group D there are indicated the ideal, and in lines 16–18 the actual values of the angle which the emergent rays make with the axis. In Groups E and F there are tabulated for the various lenses the errors of the focal length, which will require fuller consideration. Group E indicates the angular errors of the emergent rays, lines 19–21 being the values of the chromatic aberrations and lines 22–24 those of the spherical aberrations. The values of Group F indicate the corresponding linear errors, the lines 25–27 of F (1) being the chromatic aberrations, and the lines 28–30 the spherical aberrations. From the figures given in the tables it will be evident that both the chromatic and the spherical aberrations diminish as the thickness of the lens increases, the minimum value being reached when the lens is a sphere.

The focal-length errors at the principal-point ends are indicated in Group F (2) of the table. Lines 31–33 represent the chromatic differences of magnification. These errors vanish when the lens is infinitely thin. They increase with the thickness up to a maximum value and then decrease until, when the lens becomes spherical, the error is again infinitely small.

The distortion errors tabulated in lines 34–37 diminish as the thickness of the lens increases, and become zero when the lens is spherical. To find the true effect of the thickness on the distortion, it is necessary to consider only the results obtained from the yellow rays, since those for the F ray are combined with errors of chromatic differences of magnification.

The principal results obtainable from the table may be summarized in the following simple rules:—

1. While the errors at the focus ends of the focal lengths, within the given limits of thickness, do not vary by half their maximum value, the errors at the principal-point ends of the focal lengths vary through their whole maximum value, since at both extremes they are zero.
2. A sphere has the smallest focal-length errors.
3. Chromatic aberration, spherical aberration and distortion diminish regularly with the thickness until the lens becomes a sphere.
4. For an infinitely thin lens and also for a sphere the chromatic difference of magnification is zero. It increases with the thickness of the lens until in the example computed it reaches a maximum value when d = 88 mm. It then decreases and becomes zero when the lens is spherical.

II. THE INFLUENCE OF THE LENS-THICKNESS ON THE ERRORS OF THE FOCAL LENGTH

	$d_1 = 0.001$ mm. $r_0 = +69.250$ $r_2 = -225.196$ $\delta_0 = \delta_2 = 0.5$		$d_1 = 8.0$ mm. $r_0 = +69.250$ $r_2 = -216{,}195$ $\delta_0 = \delta_2 = 0.5$		$d_1 = 48.0$ mm. $r_0 = +69.250$ $r_2 = -171.153$ $\delta_0 = \delta_2 = 0.5$		$d_1 = 88.0$ mm. $r_0 = +69.250$ $r_2 = -126.115\,5$ $\delta_0 = \delta_2 = 0.5$		$d_1 = 138.5$ mm. $r_0 = +69.250$ $r_2 = -69.250$ $\delta_0 = \delta_2 = 0.5$	
	D.	F.	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·000	98·744	100·000	98·751	100·000	98·803	100·000	98·896	100·000	99·179
10			98·701	97·454	98·723	97·535	98·780	97·679	98·937	98·121
15			97·049	95·803	97·116	95·922	97·232	96·132	97·595	96·783
B. Back focal lengths: a_2':										
0·000 485	99·996	98·741	96·001	94·767	76·002	74·890	56·002	55·023	30·750	29·929
10			94·975	93·745	75·116	74·013	55·261	54·290	30·190	29·378
15			93·659	92·427	73·990	72·892	54·320	53·354	29·492	28·688
C. Abscissæ of principal points:										
1. Ideal values:										
$e' = \text{Ax.}(a_2' - f_{0,2})$:										
$e^* = e' + 0{\cdot}000$; 0·000 485	−0·004		−3·999		−23·998		−43·998		−69·250	
$e' + 0{\cdot}496$; 10			−3·503		−23·502		−43·502		−68·754	
$e' + 1{\cdot}106$; 15			−2·893		−22·892		−42·892		−68·144	
2. Actual values:										
$e = a_2' - f_{0,2}$:										
0·000 485	−0·004	−0·003	−3·999	−3·984	−23·998	−23·913	−43·998	−43·873	−69·250	−69·250
10			−3·726	−3·709	−23·607	−23·522	−43·519	−43·389	−68·747	−68·743
15			−3·390	−3·376	−23·126	−23·030	−42·912	−42·778	−68·103	−68·095
D. Semi angular aperture:										
1. Ideal values: $\beta'_{0,2}$:										
0·000 485	1·000″		1·000″		1·000″		1·000″		1·000″	
10			20 661·000		20 661·000		20 661·000		20 661·000	
15			31 056·900		31 056·900		31 056·900		31 056·900	

2. Actual values: $\beta_{0,2}$:										
0·000 485	1·000″	1·013″	1·000″	1·013″	1·000″	1·012″	1·000″	1·011″	1·000″	1·008″
10			20 827·000	210 91·500	20 821·000	21 074·000	20 810·300	21 043·500	20 777·600	20 949·200
15			31 630·200	32 034·900	31 608·800	31 995·500	31 571·400	31 927·200	31 456·000	31 716·000
E. Angular errors:										
1. Chromatic aberration:										
0·000 485	0·013″		0·013″		0·012″		0·011″		0·008″	
10			264·500		253·000		233·200		171·600	
15			404·700		386·700		355·800		260·000	
2. Spherical aberration:										
0·000 485	0·000″	0·000″	0·000″	0·013″	0·000″	0·012″	0·000″	0·011″	0·000″	0·008″
10			166·000	430·500	160·000	413·000	149·300	382·500	116·600	288·200
15			573·300	978·000	551·900	938·600	514·500	870·300	399·100	659·100
F. Errors of the focal length:										
1. At the focal point:										
(*a*) Chromatic aberration: $_{D}a_2' - {}_{F}a_2'$:	mm.		mm.		mm.		mm.		mm.	
0·000 485	+1·255		+1·234		+1·112		+0·979		+0·821	
10			+1·230		+1·103		+0·971		+0·812	
15			+1·232		+1·098		+0·966		+0·804	
(*b*) Spherical aberration: $_{\mathrm{Ax.}}a_2' - {}_{\mathrm{Rim}}a_2'$:	mm.		mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000		0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10			+1·026	+1·022	+0·886	+0·877	+0·741	+0·723	+0·560	+0·551
15			+2·342	+2·340	+2·012	+1·998	+1·682	+1·669	+1·258	+1·241
2. At the principal point:										
(*a*) Chromatic difference of magnification $_{D}e - {}_{F}e$:	mm.		mm.		mm.		mm.		mm.	
0·000 485	−0.001		−0·015		−0·085		−0·125		0·000	
10			−0·017		−0·085		−0·130		−0·005	
15			−0·014		−0·096		−0·134		−0·008	
(*b*) Distortion: $e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	−0·001	0·000	−0·015	0·000	−0·085	0·000	−0·125	0·000	0·000
10			+0·223	+0·206	+0·105	+0·020	+0·017	−0·113	−0·006	−0·011
15			+0·497	+0·483	+0·234	+0·138	+0·020	−0·114	−0·040	−0·048

The Influence of the Lens Form on the Errors of the Focal Length

In Table III (p. 112) there are summarized the computed errors in the focal length for sixteen different lenses constructed of the same glass and having the same true focal length. The sixteen lenses are so chosen that the distribution of the refraction at the two lens surfaces is different in all cases. The distribution of the deviation at the two lens surfaces is indicated by the chosen values of δ_0 and δ_2.

In the case of the first lens the value of δ_0 is greatest, namely $+2 \cdot 3$. Since the total deviation is specified to be $+1$, δ_0 must accordingly be $-1 \cdot 3$. In such a lens the first surface deviates the ray *towards* the axis, whereas the second surface deviates it *from* the axis. In the subsequent examples the absolute value of δ_0 continually decreases, that is in each succeeding lens the deviation of a yellow axial ray at the first surface *towards* the axis becomes gradually smaller, while the deviation at the second surface *from* the axis decreases by the same amount, until, in the fifth example, where $\delta_0 = 1$, the whole deviation takes place at the first surface and none at the second. Thereafter the deviation δ_0 at the first surface *towards* the axis decreases progressively until the value zero is reached, while the deviation at the second surface also *towards* the axis correspondingly increases. In the thirteenth example, where the deviation δ_0 at the first surface is zero, the total deviation *towards* the axis takes place at the second surface. In the last three examples, in which δ_0 is negative and in which its absolute value increases by stages up to $1 \cdot 2$ in the case of the last lens, the deviation at the first surface is *from* the axis and at the second surface correspondingly more *towards* the axis.

In the whole series of sixteen lenses there is an increasing transfer from the first surface to the second surface of the deviation *towards* the axis. In consequence, the lenses of the series, which commence with a meniscus having its *convex* side directed towards the incident light, progressively alter in form. In the subsequent examples the curvature of this convex face diminishes until in the thirteenth example it is plane. Thereafter the surface directed towards the incident light becomes *concave*, the refraction of the second surface being such that the total deviation of the lens is of the prescribed amount.

Thus the first five examples form a series of progressively flatter meniscus lenses, having their convex sides facing the incident light. The sixth example is a convexo-plane lens, the seventh to the twelfth inclusive are bi-convex lenses, the thirteenth is a plano-convex lens, and the fourteenth to the sixteenth inclusive are menisci having their concave surfaces facing the incident light.

The several groups A to D contain the same results of computation as the corresponding groups of the preceding table. Groups E and F indicate in the same manner the aberrations of the focal length at both the principal and focal-point ends. If the angular errors are compared, it will be seen that the aberrations at the focal points reach approximately their minimum when the refraction is divided equally over the two surfaces. This is not rigidly the case, because the distribution of the refraction at the rim is not the same as at the axis. According to the table the minimum chromatic aberration, corresponding with a semi-aperture of 15 mm., is $-405''$. This error increases in both directions to the value of $-1028''$ when the first refraction is strongest, and to $-2814''$ when the last refraction is greatest.

For the same aperture the minimum value of the spherical aberration is $-523''$. This error increases in the same way in both directions, the increase being greater the more unequal the distribution of the refraction at the two surfaces. In the first example it amounts to $-111434''$, and in the last example to $-63524''$. When these aberrations are expressed in linear form, as in groups F (1) and (2), the results, at least for the rim rays, are misleading.

In the case of the lens having equal deviations at the two surfaces the chromatic aberration for the axial ray is 1·234 mm. This error increases in both directions when the refractions are distributed unequally, but in the examples given only by an amount equal to about one quarter of the minimum value, namely to 1·557 mm. and 1·519 mm. respectively.

In the case of the rim rays, on the other hand, when the deviation is spread equally over the two surfaces, the chromatic aberration expressed in linear form differs but little for the various apertures. For the semi-aperture of 15 mm. the error is 1·232 mm., but when the distribution of the refraction over the two surfaces is altered the aberration diminishes in both directions. In the first example the error is 0·178 mm. and in the last 1·074 mm.

Similarly, the spherical aberration, the minimum value of which is reached when the refractions at the two surfaces are equal, has an error of 2·342 mm. for the semi-aperture of 15 mm., and in the limiting cases the errors are 57·938 mm. and 84·948 mm. respectively. Thus it will be seen that the variation is much less apparent than is the case when the aberrations are expressed in angular measure.

The chromatic difference of magnification reaches its minimum value when the deviation at the first surface is zero, the whole refraction then taking place at the second surface, and the example conforms to the case of a lens of no thickness in which the error is zero. For other distributions of the refraction the errors for the axial rays increase in both directions but in opposite senses. In the first instance they reach the value −0·167 mm. and in the last +0·060 mm.

The chromatic difference of magnification for the rim rays also reaches its zero value in the case mentioned above as giving the minimum for the axial ray. It varies but little when the distribution of refraction is altered. This result is due principally to the different effect of the thickness as the form of the lens varies.

The distortion is approximately zero when the deviations at the two surfaces are equal. It increases in both directions but in opposite senses, the limiting value in the first example being −22·856 mm. and in the last +16·436 mm.

It will be seen that the errors of the focal lengths at the focal points are all of the same sense, whereas the errors at the principal points change their sign. In both cases the aberrations become greater the more unequal is the distribution of refraction at the two lens surfaces.

III. THE INFLUENCE OF THE LENS FORM ON THE IMAGE ABERRATIONS

	$d_1 = 8$: $r_0 = +15.054$, $r_2 = +17.159$ $\delta_0 = +2.3$; $\delta_2 = -1.3$		$d_1 = 8$: $r_0 = +17.3125$, $r_2 = +21.605$ $\delta_0 = +2.0$; $\delta_2 = -1.0$		$d_1 = 8$: $r_0 = +19.565$, $r_2 = +26.635$ $\delta_0 = +1.77$; $\delta_2 = -0.77$		$d_1 = 8$: $r_0 = +22.612$, $r_2 = +34.625$ $\delta_0 = +1.531$; $\delta_2 = -0.531$.	
	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·044	98·320	99·998	98·419	99·995	98·510	99·998	98·587
10	63·570	62·643	74·034	72·979	80·898	79·781	86·842	85·670
15	18·144	17·961	43·614	43·103	57·875	57·160	70·544	69·651
B. Back focal lengths: a_2':								
0·000 485	81·635	80·078	83·998	82·542	85·839	84·447	87·748	86·409
10	54·280	53·433	64·480	63·501	71·595	70·541	78·140	77·025
15	23·697	23·519	42·796	42·309	55·036	54·353	66·531	65·673
C. Abscissæ of principal points:								
1. Ideal values:								
$e' = {}_{\text{Axis}}(a_2' - f_{0,2})$:								
$e^* = e' + 0{\cdot}000$ 0·000 485	−18·409	−18·409	−16·000	−16·000	−14·156	−14·156	−12·250	−12·250
$= e' + 0{\cdot}496$ 10	−17·913	−17·913	−15·504	−15·504	−13·660	−13·660	−11·754	−11·754
$= e' + 1{\cdot}106$ 15	−17·303	−17·303	−14·894	−14·894	−13·050	−13·050	−11·144	−11·144
2. Actual values:								
$e = a_2' - f_{0,2}$:								
0·000 485	−18·409	−18·242	−16·000	−15·877	−14·156	−14·063	−12·250	−12·178
10	− 9·290	− 9·210	− 9·554	− 9·478	− 9·303	− 9·240	− 8·702	− 8·645
15	+ 5·553	+ 5·558	− 0·818	− 0·794	− 2·839	− 2·807	− 4·013	− 3·978
D. Semi angular aperture:								
1. Ideal values: $\beta'_{0,2}$:								
0·000 485	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″
10	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000
15	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900

2. Actual values: $\beta_{0,2}$:								
0·000 485	1·000″	1·017″	1·000″	1·016″	1·000″	1·015″	1·000″	1·014″
10	32 183·000	32 652·000	27 693·500	28 088·500	25 368·000	25 720·000	23 648·000	23 968·000
15	142 491·000	143 519·000	68 325·000	69 077·000	52 309·000	52 935·000	43 215·000	43 752·000
E. Angular errors:								
1. Chromatic aberration:								
${}_D\beta_{0,2} - {}_F\beta_{0,2}$:								
0·000 485	− 0·017″		− 0·016″		− 0·015″		− 0·014″	
10	− 469·000		−395·000		−352·000		−320·000	
15	−1 028·000		−752·000		−626·000		−537·000	
2. Spherical aberration:								
$\beta'_{0,2} - \beta_{0,2}$: 0·000 485	0·000″	− 0·017″	0·000″	− 0·016″	0·000″	− 0·015″	0·000″	− 0·014″
10	− 11 522·000	− 11 991·000	− 7 032·500	− 7 427·500	− 4 707·000	− 5 059·000	− 2 987·000	− 3 307·000
15	−111 434·100	−112 462·100	−37 268·100	−38 020·100	−21 252·100	−21 878·100	−12 158·100	−12 695·100
F. Errors of the focal length:								
1. At the focal point:								
(*a*) Chromatic aberration:	mm.		mm.		mm.		mm.	
${}_D a_2' - {}_F a_2'$: 0·000 485	+1·557		+1·456		+1·392		+1·339	
10	+0·847		+0·979		+1·054		+1·115	
15	+0·178		+0·487		+0·683		+0·858	
(*b*) Spherical aberration:								
Axis a_2' − Rim a_2':	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10	+27·355	+26·645	+19·518	+19·041	+14·244	+13·906	+ 9·608	+ 9·384
15	+57·938	+56·559	+41·202	+40·233	+30·803	+30·094	+21·217	+20·736
2. At the principal point:								
(*a*) Chromatic difference of magnification: ${}_D e - {}_F e$:	mm.		mm.		mm.		mm.	
0·000 485	−0·167		−0·123		−0·093		−0·072	
10	−0·080		−0·076		−0·063		−0·057	
15	−0·005		−0·024		−0·032		−0·035	
(*b*) Distortion:								
$e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	− 0·167	0·000	− 0·123	0·000	− 0·093	0·000	−0·072
10	− 8·623	− 8·703	− 5·950	− 6·026	− 4·357	− 4·420	−3·052	−3·109
15	−22·856	−22·861	−14·076	−14·100	−10·211	−10·243	−7·131	−7·166

[*Continued over*

III. THE INFLUENCE OF THE LENS FORM ON THE IMAGE ABERRATIONS—*Continued*

	$d_1 = 8$: $r_0 = +34.625$, $r_2 = +92.000$ $\delta_0 = +1.0$; $\delta_2 = 0.0$		$d_1 = 8$: $r_0 = +52.966$, $r_2 = \infty$ $\delta_0 = +0.654$; $\delta_2 = +0.346$		$d_1 = 8$: $r_0 = +\ 53.000$, $r_2 = -72\,748.000$ $\delta_0 = +0.653$; $\delta_2 = +0.347$		$d_1 = 8$: $r_0 = +\ 57.708$, $r_2 = -613.282$ $\delta_0 = +0.6$: $\delta_2 = +0.4$	
	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·000	98·703	100·000	98·746	100·000	98·742	100·000	98·745
10	95·708	94·478	98·324	97·080	98·320	97·080	98·506	97·260
15	90·213	89·068	96·179	94·953	96·175	94·951	96·603	95·369
B. Back focal lengths: a_2':								
0·000 485	92·000	90·742	94·774	93·538	94·772	93·538	95·202	93·971
10	89·256	88·059	93·730	92·508	93·727	92·507	94·213	92·987
15	85·768	84·647	92·393	91·186	92·390	91·183	92·938	91·736
C. Abscissæ of principal points:								
1. Ideal values:								
$e' = {}_{\text{Axis}}(a_2' - f_{0,2})$:								
$e^* = e' + 0{\cdot}000$ 0·000 485	−8·000	−8·000	−5·231	−5·231	−5·228	−5·228	−4·798	−4·798
$= e' + 0{\cdot}496$ 10	−7·504	−7·504	−4·735	−4·735	−4·732	−4·732	−4·302	−4·302
$= e' + 1{\cdot}106$ 15	−6·894	−6·894	−4·125	−4·125	−4·122	−4·122	−3·692	−3·692
2. Actual values:								
$e = a_2' - f_{0,2}$:								
0·000 485	−8·000	−7·961	−5·231	−5·208	−5·228	−5·204	−4·798	−4·774
10	−6·452	−6·419	−4·594	−4·572	−4·593	−4·573	−4·293	−4·273
15	−4·445	−4·421	−3·786	−3·767	−3·785	−3·768	−3·665	−3·633
D. Semi angular aperture:								
1. Ideal values: $\beta'_{0,2}$:								
0·000 485	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″
10	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000
15	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900
2. Actual values: $\beta_{0,2}$:								
0·000 485	1·000″	1·013″	1·000″	1·013″	1·000″	1·012″	1·000″	1·013″
10	21 473·600	21 750·900	20 906·400	21 172·300	20 907·000	21 172·000	20 867·700	21 133·400
15	33 985·000	34 414·000	31 912·000	32 317·000	31 913·000	32 318·000	31 774·000	32 178·300

E. Angular errors:								
1. Chromatic aberration:								
$_{D}\beta_{0,2} - {_F}\beta_{0,2}$:								
0·000 485	− 0·013″		− 0·013″		− 0·012″		− 0·013″	
10	−277·300		−265·900		−265·000		−265·700	
15	−429·000		−405·000		−405·000		−404·300	
2. Spherical aberration:								
$\beta'_{0,2} - \beta_{0,2}$:								
0·000 485	0·000″	− 0·013″	0·000″	− 0·013″	0·000″	− 0·012″	0·000″	− 0·013″
10	− 812·600	−1 089·900	−245·400	− 511·300	−246·000	− 511·000	−206·700	− 472·400
15	−2 928·100	−3 357·100	−855·100	−1 260·100	−856·100	−1 261·100	−717·100	−1 121·400
F. Errors of the focal length:								
1. At the focal point:								
(*a*) Chromatic aberration:								
$_{D}a_2' - {_F}a_2'$:	mm.		mm.		mm.		mm.	
0·000 485	+1·258		+1·236		+1·234		+1·231	
10	+1·197		+1·222		+1·220		+1·226	
15	+1·121		+1·207		+1·207		+1·202	
(*b*) Spherical aberration:								
Axis a_2' − Rim a_2':	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10	+2·744	+2·683	+1·044	+1·030	+1·045	+1·031	+0·989	+0·984
15	+6·232	+6·095	+2·381	+2·352	+2·382	+2·355	+2·264	+2·235
2. At the principal point:								
(*a*) Chromatic difference of magnification $_{D}e - {_F}e$:	mm.		mm.		mm.		mm.	
0·000 485	−0·039		−0·023		−0·024		−0·024	
10	−0·033		−0·022		−0·020		−0·020	
15	−0·024		−0·019		−0·017		−0·032	
(*b*) Distortion:								
$e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	−0·039	0·000	−0·023	0·000	−0·024	0·000	−0·024
10	−1·052	−1·085	−0·141	−0·163	−0·139	−0·159	−0·009	−0·029
15	−2·449	−2·473	−0·339	−0·358	−0·337	−0·354	−0·027	−0·059

[*Continued over*

III. THE INFLUENCE OF THE LENS FORM ON THE IMAGE ABERRATIONS—*Continued*

	$d_1 = 8$: $r_0 = +69.250$, $r_2 = -216.195$ $\delta_0 = +0.5$; $\delta_2 = +0.5$		$d_1 = 8$: $r_0 = +104.529$, $r_2 = -104.529$ $\delta_0 = +0.331$; $\delta_2 = +0.669$		$d_1 = 8$: $r_0 = +216.195$, $r_2 = -69.250$ $\delta_0 = +0.160$; $\delta_2 = +0.840$		$d_1 = 8$: $r_0 = +72\,748.000$, $r_2 = -53.000$ $\delta_0 = +0.000\,5$; $\delta_2 = +0.999\,5$	
	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·000	98·751	100·005	98·760	100·000	98·751	99·999	98·742
10	98·701	97·454	98·585	97·337	97·884	96·636	96·743	95·487
15	97·049	95·803	96·770	95·519	95·141	93·901	92·412	91·174
B. Back focal lengths: a_2':								
0·000 485	96·001	94·767	97·354	96·121	98·719	97·475	99·995	98·737
10	94·975	93·745	95·812	94·575	96·064	94·824	95·765	94·520
15	93·659	92·427	93·834	92·591	92·629	91·392	90·252	89·030
C. Abscissæ of principal points:								
1. Ideal values:								
$e' = \text{Axis}(a_2' - f_{0,2})$								
$e^* = e' + 0·000$ 0·000 485	−3·999	−3·999	−2·651	−2·651	−1·281	−1·281	−0·004	−0·004
$= e' + 0·496$ 10	−3·503	−3·503	−2·155	−2·155	−0·785	−0·785	+0·492	+0·492
$= e' + 1·106$ 15	−2·893	−2·893	−1·545	−1·545	−0·175	−0·175	+1·102	+1·102
2. Actual values:								
$e = a_2' - f_{0,2}$:								
0·000 485	−3·999	−3·984	−2·651	−2·639	−1·281	−1·276	−0·004	−0·005
10	−3·726	−3·709	−2·773	−2·762	−1·820	−1·812	−0·978	−0·967
15	−3·390	−3·376	−2·936	−2·928	−2·512	−2·509	−2·160	−2·144
D. Semi angular aperture:								
1. Ideal values: $\beta'_{0,2}$:								
0·000 485	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″
10	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000
15	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900
2. Actual values: $\beta_{0,2}$:								
0·000 485	1·000″	1·013″	1·000″	1·013″	1·000″	1·013″	1·000″	1·012″
10	20 827·000	21 096·400	20 851·300	21 116·400	20 999·600	21 268·600	21 245·500	21 522·900
15	31 630·000	32 035·000	31 720·000	32 129·000	32 254·000	32 673·000	33 190·200	33 625·600

E. Angular errors:								
1. Chromatic aberration:								
$_D\beta_{0,2} - {}_F\beta_{0,2}$:								
0·000 485	− 0·013″		− 0·013″		− 0·013″		− 0·012″	
10	−269·400		−265·100		−269·000		−277·400	
15	−405·000		−409·000		−419·000		−435·400	
2. Spherical aberration:								
$\beta'_{0,2} - \beta_{0,2}$:								
0·000 485	0·000″	− 0·013″	0·000″	− 0·013″	0·000″	0·000″	0·000″	0·000″
10	−166·000	−435·400	−190·300	− 455·400	− 338·600	− 607·600	− 584·500	− 861·900
15	−523·100	−978·100	−663·100	−1 072·100	−1 197·100	−1 616·100	−2 133·300	−2 568·700
F. Errors of the focal length:								
1. At the focal point:								
(*a*) Chromatic aberration:								
$_D\alpha_2' - {}_F\alpha_2'$	mm.		mm.		mm.		mm.	
0·000 485	+1·234		+1·233		+1·244		+1·258	
10	+1·230		+1·237		+1·240		+1·245	
15	+1·232		+1·243		+1·237		+1·222	
(*b*) Spherical aberration:								
Axis α_2' − Rim α_2':	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10	+1·026	+1·022	+1·542	+1·546	+2·655	+2·651	+4·230	+4·217
15	+2·342	+2·340	+3·520	+3·530	+6·090	+6·083	+9·743	+9·707
2. At the principal point:								
(*a*) Chromatic difference of magnification: $_De - {}_Fe$:	mm.		mm.		mm.		mm.	
0·000 485	−0·015		−0·012		−0·005		+0·001	
10	−0·017		−0·011		−0·008		−0·011	
15	−0·014		−0·008		−0·003		−0·016	
(*b*) Distortion:								
$e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	+0·015	0·000	+0·012	0·000	+0·005	0·000	+0·001
10	+0·223	+0·206	+0·618	+0·607	+1·035	+1·027	+1·470	+1·459
15	+0·497	+0·483	+1·391	+1·383	+2·337	+2·334	+3·262	+3·246

[*Continued over*

III. THE INFLUENCE OF THE LENS FORM ON THE IMAGE ABERRATIONS—*Continuea*

	$d_1 = 8;$ $r_0 = \infty$ $r_2 = -52.966$ $\delta_0 = 0.0;$ $\delta_2 = +1.00$		$d_1 = 8;$ $r_0 = -92.000$ $r_2 = -34.625$ $\delta_0 = -0.376;$ $\delta_2 = +1.376$		$d_1 = 8;$ $r_0 = -34.625$ $r_2 = -22.612$ $\delta_0 = -1.0;$ $\delta_2 = +2.0$		$d_1 = 8;$ $r_0 = -28.854$ $r_2 = -20.472$ $\delta_0 = -1.2;$ $\delta_2 = +2.2$	
	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·005	98·746	100·000	98·705	99·995	98·587	100·005	98·546
10	96·716	95·475	91·981	90·759	77·963	76·821	71·822	70·716
15	92·417	91·195	81·409	80·275	48·092	47·203	30·387	29·371
B. Back focal lengths: a_2':								
0·000 485	100·005	98·746	103·010	101·701	107·996	106·540	109·605	108·086
10	95·764	94·523	92·854	91·619	80·738	79·555	74·762	73·604
15	90·249	89·028	79·587	78·440	44·238	43·313	24·657	23·583
C. Abscissæ of principal points:								
1. Ideal values:								
$e' = {}_{\text{Axis}}(a_2' - f_{0,2})$:								
$e^* = e' + 0{\cdot}000$ 0·000 485	0·000	0·000	+3·010	+3·010	+8·001	+8·001	+ 9·600	+ 9·600
$= e' + 0{\cdot}496$ 10	+0·496	+0·496	+3·506	+3·506	+8·497	+8·497	+10·096	+10·096
$= e' + 1{\cdot}106$ 15	+1·106	+1·106	+4·116	+4·116	+9·107	+9·107	+10·706	+10·706
2. Actual values:								
$e = a_2' - f_{0,2}$								
0·000 485	0·000	0·000	+3·010	+2·996	+8·001	+7·953	+9·600	+9·540
10	−0·952	−0·952	+0·873	+0·860	+2·775	+2·734	+2·940	+2·888
15	−2·168	−2·167	−1·822	−1·835	−3·854	−3·890	−5·730	−5·788
D. Semi angular aperture:								
1. Ideal values: $\beta'_{0,2}$:								
0·000 485	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″
10	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000
15	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900

2. Actual values: $\beta_{0,2}$:								
0·000 485	1·000″	1·013″	1·000″	1·013″	1·000″	1·014″	1·000″	1·015″
10	21 252·300	21 526·600	22 336·900	22 635·300	26 313·000	26 700·000	28 535·000	28 925·000
15	33 190·600	33 627·800	37 584·000	38 103·000	62 362·000	63 464·000	94 581·000	97 395·000
E. Angular errors:								
1. Chromatic aberration:								
$_D\beta_{0,2} - {}_F\beta_{0,2}$:								
0·000 485	− 0·013″		− 0·013″		− 0·014″		− 0·015″	
10	−274·300		−298·400		− 387·000		− 390·000	
15	−437·200		−519·000		−1 102·000		−2 814·000	
2. Spherical aberration:								
$\beta'_{0,2} - \beta_{0,2}$:								
0·000 485	0·000″	− 0·013″	0·000″	− 0·013″	0·000″	− 0·014″	0·000″	− 0·015″
10	− 591·300	− 865·600	−1 675·900	−1 974·300	− 5 652·000	− 6 039·000	− 7 874·000	− 8 264·000
15	−2 133·700	−2 570·900	−6 527·100	−7 046·100	−31 305·100	−32 407·100	−63 524·100	−66 338·100
F. Errors of the focal length:								
1. At the focal point:								
(*a*) Chromatic aberration:								
$_Da_2' - {}_Fa_2'$:	mm.		mm.		mm.		mm.	
0·000 485	+1·259		+1·309		+1·456		+1·519	
10	+1·241		+1·235		+1·183		+1·158	
15	+1·221		+1·147		+0·925		+1·074	
(*b*) Spherical aberration:								
Axis a_2' − Rim a_2':	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10	+4·241	+4·223	+10·156	+10·082	+27·258	+26·985	+34·843	+34·482
15	+9·756	+9·718	+23·423	+23·261	+63·758	+63·227	+84·948	+84·503
2. At the principal point:								
(*a*) Chromatic difference of magnification: $_De - {}_Fe$:	mm.		mm.		mm.		mm.	
0·000 485	0·000		+0·014		+0·048		+0·060	
10	0·000		+0·013		+0·041		+0·052	
15	−0·001		+0·013		+0·036		+0·058	
(*b*) Distortion:								
$e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	+0·014	0·000	+ 0·048	0·000	+ 0·060
10	+1·448	+1·448	+2·633	+2·646	+ 5·722	+ 5·763	+ 7·156	+ 7·208
15	+3·274	+3·273	+5·938	+5·951	+12·961	+12·997	+16·436	+16·494

The Influence on the Focal-length Errors of the Type of Glass chosen

In Table IV, pp. 122–4, there are given the results of computation for four lenses made of different types of glasses but having the same power, thickness, and the form in which the refractions at the two lens surfaces are equal. The results are tabulated under Groups A to D in precisely the same way as those of the previous tables, the incidence heights being as before, but in addition there are included the results for an incidence height of 22·5 mm.

Glasses used for the construction of lenses may be distinguished by the values of their refractive indices for D rays on the one hand and by the values of their dispersions on the other hand. In the examples chosen both values are greater in each subsequent lens of the series. If the refraction or dispersion of the selected type of glass alters, the position of the focus end of the focal length is appreciably affected, while the principal-point end is affected to a smaller degree. If the refractive index remains constant while the dispersion is altered, there will be involved no alteration in the radii of the surfaces of lenses of equal power and thickness, but the chromatic aberration will be appreciably affected, whereas the chromatic difference of magnification will alter but little. The greater the dispersion of the selected type of glass the greater will be the chromatic aberration.

For lenses of the same power and thickness alterations of the refractive index for D rays involve an alteration of the radii of the surfaces. An increase of refractive index involves an increase of the radii and vice versa. If the radii are increased the maximum aperture becomes greater. The spherical aberration decreases in value as the refractive index of the D ray increases.

From the table it will also be seen that in the case of the glasses of weaker refraction the spherical aberration for a semi-aperture of 22.5 mm. is greater than the chromatic aberration, whereas for the glasses of strongest refraction the spherical aberration corresponding with the same aperture is smaller than the chromatic aberration. The stronger the refraction of the glass employed the greater will be the aperture at which the spherical and chromatic aberrations are equally large.

From the results tabulated in the tables the following general conclusions may be drawn:—

1. Alterations of the errors at the focus end of the focal length are effected principally by modifications of the lens radii and of the type of glass employed, it being understood that the aperture of the lens is constant.
2. The thickness of the lens is the element by means of which alterations are effected at the principal-point end of the focal length.
3. The effect of the thickness—that is, the separation of the two refracting surfaces—is the greater the more unequal is the distribution of the refraction at the two surfaces.

IV. INFLUENCE OF THE CHOSEN GLASS ON THE FOCAL-LENGTH ERRORS

	$_Dn = 1.515 9$; $_Fn = 1.521 1$ $d_1 = 8$: $r_0 = +68.064$, $r_2 = -204.621$ $\delta_0 = \delta_2 = +0.5$		$_Dn = 1.529 7$; $_Fn = 1.536 4$ $d_1 = 8$; $r_0 = +69.250$, $r_2 = -216.195$ $\delta_0 = \delta_2 = +0.5$		$_Dn = 1.625 5$; $_Fn = 1.637 9$ $d_1 = 8$: $r_0 = +76.963$, $r_2 = -320.730$ $\delta_0 = \delta_2 = +0.5$		$_Dn = 1.962 6$; $_Fn = 1.998 1$ $d_1 = 8$; $r_0 = +98.093$, $r_2 = -4 945.500$ $\delta_0 = \delta_2 = +0.5$	
	D.	F.	D.	F.	D.	F.	D.	F.
A. Focal lengths: $f_{0,2}$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	100·000	99·008	100·000	98·751	100·000	98·073	100·000	96·445
10	98·662	97·685	98·701	97·454	98·903	96·977	99·229	95·667
15	96·962	95·973	97·049	95·803	97·510	95·585	98·254	94·689
22·5	92·991	92·011	93·194	91·959	94·282	92·359	96·024	92·436
B. Back focal lengths: a_2'								
0·000 485	96·000	95·026	96·001	94·767	96·005	94·109	96·000	92·517
10	94·929	93·972	94·975	93·745	95·171	93·279	95·494	92·002
15	93·581	92·594	93·659	92·427	94·119	92·213	94·849	91·346
22·5	90·344	89·367	90·545	89·325	91·633	89·729	93·356	89·825
C. Abscissæ of principal points:								
1. Ideal values:								
$e' = {}_{\text{Axis}}(a_2' - f_{0,2})$:								
$e^* = e' + 0·000$ 0·000 485	−4·000	−4·000	−3·999	−3·999	−3·995	−3·995	−4·000	−4·000
$= e' + 0·496$ 10	−3·504	−3·504	−3·503	−3·503	−3·499	−3·499	−3·504	−3·504
$= e' + 1·106$ 15	−2·894	−2·894	−2·893	−2·883	−2·889	−2·889	−2·894	−2·894
$= e' + 2·439$ 22·5	−1·561	−1·561	−1·560	−1·560	−1·556	−1·556	−1·561	−1·561
2. Actual values:								
$e = (a_2' - f_{0,2})$								
0·000 485	−4·000	−3·982	−3·999	−3·984	−3·995	−3·964	−4·000	−3·928
10	−3·733	−3·713	−3·726	−3·709	−3·732	−3·698	−3·735	−3·665
15	−3·385	−3·379	−3·390	−3·376	−3·391	−3·372	−3·405	−3·343
22·5	−2·647	−2·644	−2·649	−2·634	−2·649	−2·630	−2·668	−2·611

D. Semi angular aperture:								
1. Ideal values: $\beta'_{0,2}$:								
0·000 485	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″	1·000″
10	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000	20 661·000
15	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900	31 056·900
22·5	46 810·300	46 810·300	46 810·300	46 810·300	46 810·300	46 810·300	46 810·300	46 810·300
2. Actual values: $\beta_{0,2}$								
0·000 485	1·000″	1·010″	1·000″	1·013″	1·000″	1·020″	1·000″	1·037″
10	20 835·000	21 042·000	20 827·000	21 091·400	20 784·400	21 194·500	20 717·000	21 483·000
15	31 658·000	31 979·000	31 630·000	32 035·000	31 483·000	32 107·000	31 248·000	32 406·000
22·5	48 966·000	49 468·000	48 863·000	49 495·000	48 320·000	49 289·000	47 475·000	49 249·000
E. Angular errors:								
1. Chromatic aberration:								
$_{D}\beta_{0,2} - {}_{F}\beta_{0,2}$:								
0·000 485	− 1·010″		− 1·013″		− 1·020″		− 1·037″	
10	− 207·000		− 264·400		− 410·100		− 766·000	
15	− 321·000		− 405·000		− 624·000		− 1 158·000	
22·5	− 502·000		− 632·000		− 969·000		− 1 774·000	
2. Spherical aberration:								
$\beta'_{0,2} - \beta_{0,2}$:								
0·000 485	0·000″	− 0·010″	0·000″	− 0·013″	0·000″	− 0·020″	0·000″	− 0·037″
10	− 174·000	− 381·000	− 166·000	− 430·400	− 123·400	− 533·500	− 56·000	− 822·000
15	− 601·100	− 922·100	− 573·100	− 978·100	− 426·100	− 1 050·100	− 191·100	− 1 349·100
22·5	− 2 155·700	− 2 657·700	− 2 052·700	− 2 684·700	− 1 509·700	− 2 478·700	− 664·700	− 2 438·700
F. Errors of the focal length:								
1. At the focal point:								
(*a*) Chromatic aberration:								
$_{D}a'_2 - {}_{F}a'_2$	mm.		mm.		mm.		mm.	
0·000 485	+0·974		+1·234		+1·896		+3·483	
10	+0·957		+1·230		+1·892		+3·492	
15	+0·987		+1·232		+1·906		+3·503	
22·5	+0·977		+1·220		+1·904		+3·531	

[Continued over

IV. INFLUENCE OF THE CHOSEN GLASS ON THE FOCAL-LENGTH ERRORS—*Continuea*

	$_{D}n = 1.5159$; $_{F}n = 1.5211$ $d_1 = 8$: $r_0 = +68.064$, $r_2 = -204.621$ $\delta_0 = \delta_2 = +0.5$		$_{D}n = 1.5297$; $_{F}n = 1.5364$ $d_1 = 8$: $r_0 = +69.250$, $r_2 = -216.195$ $\delta_0 = \delta_2 = +0.5$		$_{D}n = 1.6255$; $_{F}n = 1.6379$ $d_1 = 8$: $r_0 = +76.963$, $r_2 = -320.730$ $\delta_0 = \delta_2 = +0.5$		$_{D}n = 1.9626$; $_{F}n = 1.9981$ $d_1 = 8$: $r_0 = +98.093$, $r_2 = -4945.500$ $\delta_0 = \delta_2 = +0.5$	
	D.	F.	D.	F.	D.	F.	D.	F.
F. 1. (*b*) Spherical aberration:								
Axis a_2' − Rim a_2'	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
10	+1·071	+1·054	+1·026	+1·022	+0·834	+0·830	+0·506	+0·515
15	+2·419	+2·432	+2·342	+2·340	+1·886	+1·896	+1·151	+1·171
22·5	+5·656	+5·659	+5·456	+5·442	+4·372	+4·380	+2·644	+2·692
2. At the principal point:								
(*a*) Chromatic difference of magnification: $_{D}e - {}_{F}e$:	mm.		mm.		mm.		mm.	
0·000 485	−0·018		−0·015		−0·031		−0·072	
10	−0·020		−0·017		−0·034		−0·070	
15	−0·006		−0·014		−0·019		−0·062	
22·5	−0·003		−0·015		−0·019		−0·057	
(*b*) Distortion:								
$e^* - e$:	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
0·000 485	0·000	−0·018	+0·000	−0·015	0·000	−0·031	0·000	−0·072
10	+0·229	+0·209	+0·223	+0·206	+0·233	+0·199	+0·231	+0·161
15	+0·491	+0·485	+0·497	+0·493	+0·502	+0·483	+0·511	+0·449
22·5	+1·086	+1·083	+1·089	+1·074	+1·093	+1·074	+1·107	+1·050

The Diameter of the Image of an Axial Object Point and the Angle subtended by it at the Second Principal Point

In the preceding sections it has been shown that the rays of light proceeding from an axial object point situated at an infinite distance from the lens do not meet the axis in one point. An image surface instead of a point image is therefore formed, and the best position of the image may be regarded as being in that plane normal to the axis (the best focus plane) in which the light appears to be most concentrated.

Gauss‡ and Bessel§ have dealt with this subject of the best choice of the focus plane, the former in a letter to Brandt, and the latter in his investigation of the Königsberg heliometer. Both state that considerable latitude is possible in the choice, but that it is advisable to adopt a method based upon the principle of least squares. From this it follows that the greatest concentration of the light will take place at a plane normal to the axis where the product of the quantity of the light forming a particular image point and the square of its distance from the axis is a minimum.

Bessel has evolved the following formula for the determination of the distance of the best focus plane from the vertex of the last refracting surface:—

$$(94) \quad \frac{7(a_2'k)_0 + 32(a_2'k)_{3\cdot75} + 12(a_2'k)_{7\cdot5} + 32(a_2'k)_{10\cdot25} + 7(a_2'k)_{15}}{7(k)_0 + 32(k)_{3\cdot75} + 12(k)_{7\cdot5} + 32(k)_{10\cdot25} + 7(k)_{15}},$$

where the value of k is $(\tan^2 \beta_{0,2} \cos \eta_0 \sin \eta_0)$ for the respective incidence heights, and where $\beta_{0,2}$ is the angle which the emergent rays make with the axis, a_2' is the distance to the point of intersection of these rays with the axis measured outwards from the vertex of the last refracting surface, and η_0 is the angle subtended by the incidence height at the centre of the first surface. The suffixes at the right-hand side of the brackets indicate the incidence height which has to be taken in evaluating the terms within the particular bracket.

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‡ Gehler's *Physikalisches Wörterbuch*, new edition, Leipzig, 1831, Vol. VI., Pt. I, p. 437.

§ Bessel's *Astronomische Untersuchungen*, Königsberg, 1841, Vol. I, p. 103.

As an example, the calculation will be made for the lens of Table III, in which the refraction at the two surfaces is equal, and in which the values of h_0, $\beta_{0,\,2}$, η_0, and a_2' are as follows:—

h_0.	$\beta_{0,\,2}$.	η_0.	a_2'.
0·0	0° 0′ 0·0″	0° 0′ 0·0″	96·000 mm.
3·75	2 9 5·5	3 6 15·0	95·857 ,,
7·5	4 19 13·9	6 13 3·0	95·426 ,,
10·25	5 55 58·1	8 30 42·4	94·925 ,,
15·0	8 47 10·0	12 30 35·0	93·659 ,,

By means of the above-mentioned formula it will be found that the distance of the best focus plane from the last refracting surface is 94·517 mm., and that the chosen rays intersect this plane at heights above the axis of:

mm.	mm.	mm.	mm.		mm.
0·0000,	+0·0503,	+0·0687,	+0·0424,	and	−0·1326,

and that these various heights of intersection subtend at the second principal point the following angles, viz.:—

+0·00″, +106″, +144″, +89″, and −278″.

In a similar manner the best focus plane should be determined for the remaining lenses, in order that the character of the diffuse images formed in these planes may be discussed. These computations entail a disproportionate expenditure of time, and accordingly another, although less exact, method will be employed. The method is all the more suitable considering that in the case of a single lens the aberrations of the image are large, and that in consequence no great accuracy is required. It is of first importance that in similar calculations the same method should always be used for the determination of the best focus-plane position, and from it the diameter of the image of a point, since in comparative computations for the determination of the best arrangement it is the relative rather than the absolute values which are of importance.

Gauss‡ has stated that in the case of a lens combination, if the axial and rim rays meet the axis at the same position, the ray incident at a height of two-thirds the semi-aperture exhibits the greatest residual aberration, and that in this case the plane which passes through the point of intersection of the emergent lower rim ray with the emergent upper two-thirds rim ray may be regarded as the plane of best focus. This rule will now be applied to the case of an image produced by a single lens, although in this instance the axial and peripheral rays intersect the axis at different places.

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‡ "On the Achromatic Doublet Objective, particularly with regard to the complete elimination of colour dispersion", by F. Gauss, *Zeitschriften für Astronomie*, by Bohnenberger and Lindenau, Tübingen, 1817, Vol. IV, p. 345.

Suppose ①[‡] and ② represent the distances from the points of intersectioh with tne axis of the lower rim ray and of the upper two-thirds rim ray, respectively, to the intersection of the best focus plane with the axis, both dimensions being positive when measured in the direction of the incident light; then

$$\text{(95)} \qquad \text{①} - \text{②} = {}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2' ;$$

and, further,

$$\text{(96)} \qquad h' = \text{①} \tan\left({}_{\text{rim}}\beta_{0,\,2}\right),$$

and

$$\text{(97)}[\S] \qquad -h' = \text{②} \tan\left({}_{\frac{2}{3}\text{-rim}}\beta_{0,\,2}\right).$$

From equations 95, 96, and 97 the values of ①, ②, and h' may be determined thus:—

$$\text{(98)} \qquad \text{②} = -\frac{{}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2'}{1 + \dfrac{\tan\left({}_{\frac{2}{3}\text{-rim}}\beta_{0,\,2}\right)}{\tan\left({}_{\text{rim}}\beta_{0,\,2}\right)}}.$$

$$\text{(99)} \qquad \text{①} = \frac{{}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2'}{1 + \dfrac{\tan\left({}_{\text{rim}}\beta_{0,\,2}\right)}{\tan\left({}_{\frac{2}{3}\text{-rim}}\beta_{0,\,2}\right)}}.$$

$$\text{(100)} \qquad h' = \text{①} \tan\left({}_{\text{rim}}\beta_{0,\,2}\right) = -\,\text{②} \tan\left({}_{\frac{2}{3}\text{-rim}}\beta_{0,\,2}\right).$$

The distance of the best focus plane from the last refracting surface is then the value of

$$\text{(101)} \qquad {}_{\text{rim}}a_2' + \text{①} = {}_{\frac{2}{3}\text{-rim}}a_2' + \text{②}.$$

In this calculation the rim and two-thirds rays may be reckoned for the D line or for the F line. Thus two different positions of the best focus plane corresponding with the two colours are obtained. When the light comprises both colours there is a third focus plane which contains the resultant image. Its position is determined by the point of intersection of the lower rim ray of the stronger refraction with the upper two-thirds rim ray of the lesser refraction. The corresponding value of h' is the radius of the image circle in the chosen focus plane.‡

[‡] The distances in question are denoted by these conspicuous signs to avoid any possibility of confusion.

[§] In equations 96 and 97 $\beta_{0,\,2}$ represents a deviation, and has the same sign in both equations.—ED.

As an example, there will be chosen again the lens in which equal refraction takes place at both surfaces. In this case

$$\begin{aligned} {}_{15}a_2' &= 93{\cdot}659 \text{ mm.}, \\ {}_{10}a_2' &= 94{\cdot}975 \text{ mm.}, \\ \log \tan {}_{15}\beta_{0,\,2} &= 9{\cdot}189\,10; \\ \log \tan {}_{10}\beta_{0,\,2} &= 9{\cdot}005\,68. \end{aligned}$$

It follows that

$$① - ② = 94{\cdot}975 - 93{\cdot}659 = 1{\cdot}316 \text{ mm.},$$

and further,

$$\begin{aligned} \log \tan {}_{10}\beta_{0,\,2} &= 9{\cdot}005\,68 \\ \log \tan {}_{15}\beta_{0,\,2} &= \underline{9{\cdot}189\,10} \\ \log \tan {}_{10}\beta_{0,\,2} - \log \tan {}_{15}\beta_{0,\,2} &= 9{\cdot}816\,58 \\ \tan {}_{10}\beta_{0,\,2} : \tan {}_{15}\beta_{0,\,2} &= 0{\cdot}655\,51 \\ 1 &= \underline{1\phantom{{\cdot}655\,51}} \\ 1 + \frac{\tan {}_{10}\beta_{0,\,2}}{\tan {}_{15}\beta_{0,\,2}} &= 1{\cdot}655\,51 \\ \log(-1{\cdot}316) &= 0{\cdot}119\,26n \\ \log 1{\cdot}655\,51 &= \underline{0{\cdot}218\,93} \\ \log ② &= 9{\cdot}900\,33n \\ ② &= -0{\cdot}794\,93 \\ ① &= \underline{+0{\cdot}521\,07} \\ ① - ② &= +1{\cdot}316 \end{aligned}$$

---

‡ For the calculation of the focus plane there may be used approximate formulæ which lead more quickly to a final result than does the use of the above-mentioned formulæ. Thus, the distance of the focus plane from the vertex of the last refracting substance may be obtained from the equation

$$\text{(102)} \qquad {}_{\text{rim}}a_2' + ① = {}_{\frac{2}{3}\text{-rim}}a_2' + ② = {}_{\text{rim}}a_2' + 0.4({}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2').$$

This method will not be used here, since in a single lens the aberrations that exist are so great that it is advisable to use the more exact method of reckoning. In the second volume, where lens combinations are considered, this approximate method will again be referred to.

$$\log \text{①} = 9{\cdot}716\,90$$
$$\log \tan_{15}\beta_{0,2} = 9{\cdot}189\,10$$
$$\log h' = 8{\cdot}906\,00$$

$$h' = 0{\cdot}080\,54$$
$${}_{15}a_2' = 93{\cdot}659$$
$$+\text{①} = 0{\cdot}521$$

Distance of best focus plane measured from the vertex of the last surface ... = 94·180[‡]

$$-e = 3{\cdot}999$$

Distance of best focus plane measured from second principal point ... ... = 98·179

It is now possible to calculate the angle which the image diameter subtends at the principal point. Suppose the value of half this angle is μ, then from fig. 54 it will be seen that

$$\text{(103)} \qquad \tan \mu = \frac{h'}{{}_{\text{rim}}a_2' + \text{①} - e}$$

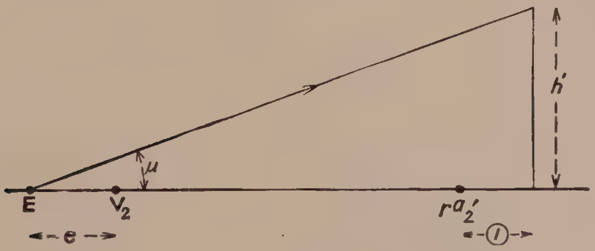

Fig. 54

from which μ may be determined and the value of the desired angle, namely, that subtended by the image diameter at the principal point, be found by multiplying by two.

[‡] From the above-mentioned less-accurate formulæ (102),

$$0.4({}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2') = 0.4 \times 1.316 = 0.526,$$

and

$${}_{\text{rim}}a_2' = 93.659$$
$$0.4({}_{\frac{2}{3}\text{-rim}}a_2' - {}_{\text{rim}}a_2') = 0.526$$

Distance to best focus plane measured from vertex of last refracting surface $\doteq 94.185$

This agrees very closely with the more accurate value.

For the above example,

$$ {}_{15}a_2' + \textcircled{1} - e = 98{\cdot}179 $$

and

$$ \begin{aligned} \log h' &= 8{\cdot}906\,00 \\ \log ({}_{15}a_2' + \textcircled{1} - e) &= \underline{1{\cdot}992\,02} \\ \log \tan \mu &= 6{\cdot}913\,98 \\ \mu &= 0^\circ\ 2'\ 50{\cdot}5'' \\ \text{and } 2\mu &= 0^\circ\ 5'\ 41''. \end{aligned} $$

It will be evident that the angle 2μ is a measure of the perfection of the lens design. If, for instance, the focal length and the aperture are constant, then the diameter of the image of a point, or alternatively the angle which it subtends at the second principal point, is a measure of the clearness of definition that is realized.

If the power of the system is altered (by altering the focal length), while the form remains the same, the diameter of the image of a point alters, while the angle under which it appears when viewed from the second principal point remains unchanged. The angle 2μ is accordingly the element by means of which the perfection of the design is expressed, quite independently of the power, whereas the diameter of the image of a point depends upon the lens form and the power.

In the following tables the calculated values of the diameter of the image and of the angle 2μ for the various lenses are indicated.

1. Table V contains the results for a lens of focal length 100 mm., in which the refraction is equally distributed over the two surfaces. It illustrates the influence of the lens aperture on the image diameter and subtended angle.

The table comprises four sections: the first gives the semi-aperture of the lens; the second the distance of the best focus plane from the principal point; the third the diameter of the image of a point measured in the best focus plane; and the fourth the angle which the image diameter subtends at the principal point.

As already described, the best focus plane is assumed to be that plane normal to the axis which contains the point of intersection of the *lower* rim ray with the *upper two-thirds* rim ray.

Each of the last three sections is divided into three columns:

V. INFLUENCE OF THE LENS APERTURE UPON THE DIAMETER OF THE IMAGE AND UPON THE ANGLE IT SUBTENDS

Semi-Aperture.	Distance of Best Focus Plane from Principal Point.			Diameter (2h′) of Image.			Image Angle subtended at Second Principal Point.		
	D.	F.	D. and F.	D.	F.	D. and F.	D.	F.	D. and F.
	mm.	mm.	mm.	mm.	mm.	mm.			
0	100·000	98·751	99·376						
2·963	99·931	98·697	99·387	0·001 02	0·001 03	0·030 44	0° 0′ 2″	0° 0′ 2″	0° 1′ 3″
4·444	99·845	98·612	99·101	0·004 10	0·004 11	0·048 25	0 0 8	0 0 8	0 1 40
6·667	99·648	98·415	98·903	0·013 49	0·013 61	0·079 85	0 0 28	0 0 28	0 2 47
10·000	99·202	97·971	98·457	0·046 24	0·046 66	0·146 23	0 1 36	0 1 38	0 5 6
15·000	98·179	96·948	97·427	0·161 07	0·163 40	0·313 47	0 5 41	0 5 48	0 11 4
22·500	95·763	94·533	95·001	0·585 9	0·592 7	0·821 4	0 21 2	0 21 33	0 29 43

the first contains the values for light of colour D; the second those for light of colour F; and the third the values obtained when both colours are taken into account at the same time. It will be seen that the values of the diameter increase in the same proportion as the angles which they subtend at the principal point, and that both vary approximately in the same ratio as the angular values of the spherical aberration (Table I, E 2), whereas the linear values of the spherical aberration (Table I, F 2) increase much more slowly. This fully confirms the statement previously made, namely, that the angular values provide a truer idea of the influence of the aberrations on the definition of the images than is obtainable from linear values.

Further, from the table it will be seen that the values in the third columns of the sections differ as compared with those of the first and second columns by a proportionally greater amount the smaller the effective aperture. This arises from the fact that the chromatic aberration is of considerable amount even when the aperture is small, whereas the spherical aberration is almost zero.

2. In Table VI there is indicated the effect of the lens thickness on the diameter of the image and the angle which it subtends at the principal point. For this purpose the calculations are made for lenses having thicknesses of 8 mm., 48 mm., 88 mm., and 138·5 mm. (the last being a sphere), all the lenses having the same aperture, viz. 15 mm., and the same true focal length, viz. 100 mm. The results are presented in tabular form.

This table is arranged similarly to Table V. It will be seen from it that as the lens thickness increases till the form becomes spherical, the definition continually improves, the error diminishing to less than half its maximum value; and that, accordingly, for a given aperture and focal length a sphere used as a magnifying lens gives a better result than a lens of smaller thickness, at least for rays proceeding from an object point in the axis.

3. In order to illustrate the influence of the lens form upon the image diameter and upon the angle which it subtends at the principal point, these values have been calculated for various lenses having the same focal length of 100 mm., and the same aperture of 15 mm., but in which the distribution of the refractions over the two surfaces varies from

$$\delta_0 = +2{\cdot}3 \text{ and } \delta_2 = -1{\cdot}3$$

to

$$\delta_0 = -1{\cdot}2 \text{ and } \delta_2 = +2{\cdot}2.$$

VI. EFFECT OF THE LENS THICKNESS ON THE DIAMETER OF THE IMAGE OF A POINT AND UPON THE ANGLE SUBTENDED BY IT

Lens Thickness	Distance of Best Focus Plane from Principal Point.			Diameter ($2h'$) of Image.			Image Angle subtended at Principal Point.		
	D.	F.	D. and F.	D.	F.	D. and F.	D.	F.	D. and F.
mm.	mm.	mm.	mm.	mm.	mm.	mm.			
8	98·179	96·948	97·427	0·161 1	0·163 4	0·313 5	0° 5′ 41″	0° 5′ 48″	0° 11′ 4″
48	98·447	97·337	97·767	0·136 7	0·138 9	0·271 4	0 4 46	0 4 54	0 9 36
88	98·703	97·726	98·106	0·115 0	0·115 0	0·233 7	0 4 0	0 4 2	0 8 11
138·5	99·019	98·212	98·531	0·085 3	0·084 8	0·183 8	0 2 58	0 2 58	0 6 25

Table VII is arranged in the same way as the previous ones; but from the table it will now be seen that a minimum value of the aberration is reached when the refraction over the two lens faces is equal, that is when

$$\delta_0 = \delta_2 = 0{\cdot}5.$$

The errors increase rapidly in both directions, the increase being much quicker when the stronger refraction takes place at the second surface; for example, when

$$\delta_0 = +2{\cdot}0 \text{ and } \delta_2 = -1{\cdot}0,$$

the angle subtended by the diameter of the image at the principal point, when D and F rays are simultaneously taken into account, is $3^\circ\ 50'\ 18''$, whereas when

$$\delta_0 = -1 \text{ and } \delta_2 = +2,$$

the same angle has a value of $8^\circ\ 28'\ 52''$.

4. Finally, there may be determined the influence of the particular type of glass chosen for the construction of the lens on the diameter and subtended angle of the image. The calculations have been made with respect to four different types of glass. Again, the focal length of the lens is assumed to be 100 mm., and the apertures 15 mm. and 22·5 mm.

Table VIII in its arrangement is very similar to the previous ones, but only the results corresponding with semi-apertures of 15 mm. and 22·5 mm. are given.

It will be seen from the table that for monochromatic light the stronger the refracting power of the glass employed the smaller are the aberrations produced, but that in the case of light comprising the two chosen colours the errors are greater for the glasses of stronger refraction, since in the particular glasses chosen the dispersion increases in a greater proportion than does the refraction. When greater apertures are employed, the rate of increase of the resultant error is smaller. From this it may be concluded that if the refraction of the glass increases while the dispersion increases in an equal or smaller proportion, clearer definition may be obtained by the use of stronger glasses.

VII. THE INFLUENCE OF THE LENS FORM ON THE LINEAR AND ANGULAR SIZE OF THE IMAGE OF A POINT

Deviations of the Lenses.		Distance of the Best Focus Plane from the Principal Point in Millimetres.			Diameter of the Image in Millimetres.			Angle subtended by the Image at the Principal Point.		
δ_0	δ_2	D.	F.	D. and F.	D.	F.	D. and F.	D.	F.	D. and F.
+2·3	−1·3	46·995	46·728	46·803	8·083 7	8·017 9	8·143 8	9° 49′ 53″	9° 48′ 26″	9° 56′ 40″
+2·0	−1·0	64·911	64·296	64·508	4·205 9	4·167 1	4·314 8	3 42 40	3 42 44	3 49 52
+1·770	−0·770	74·539	73·742	74·030	2·771 8	2·746 7	2·897 9	2 7 50	2 8 2	2 14 33
+1·531	−0·531	82·859	81·913	82·267	1·734 3	1·718 6	1·870 9	1 11 57	1 12 8	1 18 11
+1·0	−0·0	95·114	93·964	94·412	0·447 5	0·443 5	0·594 4	0 16 11	0 16 14	0 21 39
+0·654	+0·346	98·152	96·939	97·413	0·164 6	0·164 9	0·314 8	0 5 45	0 5 50	0 11 7
+0·653	+0·347	98·146	96·934	97·407	0·164 6	0·165 1	0·314 8	0 5 45	0 5 51	0 11 7
+0·5	+0·5	98·179	96·948	97·427	0·161 1	0·163 4	0·313 5	0 5 41	0 5 48	0 11 4
+0·331	+0·669	97·268	96·027	96·506	0·242 6	0·246 4	0·397 0	0 8 34	0 8 50	0 14 8
+0·160	+0·840	95·261	94·022	94·495	0·425 9	0·431 1	0·582 3	0 15 22	0 15 46	0 21 10
+0·000 5	+0·999 5	92·401	91·170	91·634	0·696 5	0·702 4	0·855 2	0 25 56	0 26 30	0 32 6
+0·0	+1·0	92·395	91·166	91·628	0·696 6	0·703 3	0·855 3	0 25 56	0 26 32	0 32 6
−0·376	+1·376	81·500	80·319	80·732	1·814 2	1·826 9	1·981 3	1 16 31	1 18 10	1 24 22
−1·0	+2·0	46·873	45·844	46·074	6·634 9	6·693 9	6·839 8	8 5 48	8 21 5	8 29 25
−1·2	+2·2	26·080	24·829	24·946	10·882 8	11·079 2	11·198 5	23 34 19	25 9 18	25 18 6

VIII. THE INFLUENCE OF THE CHOICE OF GLASS ON THE LINEAR AND ANGULAR SIZE OF THE IMAGE OF A POINT

Refractive Index of the Glass of the Lens.		Distance of the Best Focus Plane from the Principal Point.			Diameter of the Image in Millimetres.			Angle subtended by the Image at the Principal Point.		
$_Dn_1$	$_Fn_1$	D.	F.	D. and F.	D.	F.	D. and F.	D.	F.	D. and F.
					Semi-aperture, 15 mm.					
1·515 9	1·521 1	98·115	97·139	97·513	0·165 1	0·170 5	0·287 1	0° 5′ 47″	0° 6′ 3″	0° 10′ 7″
1·529 7	1·536 4	98·179	96·948	97·427	0·161 1	0·163 4	0·313 5	0 5 41	0 5 48	0 11 4
1·625 5	1·637 9	98·531	96·631	97·367	0·128 4	0·132 7	0·363 8	0 4 29	0 4 43	0 12 51
1·962 6	1·998 1	99·106	95·607	96·959	0·078 3	0·082 6	0·511 0	0 2 42	0 2 58	0 18 7
					Semi-aperture, 22·5 mm.					
1·515 9	1·521 1	95·606	94·625	95·000	0·610 9	0·615 4	0·798 6	0° 21′ 58″	0° 22′ 22″	0° 28′ 55″
1·529 7	1·536 4	95·763	94·533	95·001	0·585 9	0·592 7	0·821 4	0 21 2	0 21 33	0 29 43
1·625 5	1·637 9	96·602	94·697	95·423	0·465 2	0·474 2	0·827 9	0 16 33	0 17 13	0 29 50
1·962 6	1·998 1	97·945	94·425	95·762	0·276 0	0·291 9	0·942 7	0 9 42	0 10 39	0 33 52

Formulæ for the Trigonometrical Computation of Rays lying out of the Axis but in the Plane containing it

On p. 59 of this chapter it is stated that for the computation of an image point lying out of the axis there will be chosen four rays, namely:—

1. The principal ray.
2. The upper rim ray.
3. The lower rim ray.
4. A ray oriented 90° with respect to 3.

I. When no Radius is very large:—

$$\left.\begin{aligned} &1.\quad \sin\phi_0 = \frac{(e_{2,0} - r_0)\sin\tau_0}{r_0}.\\ &2.\quad \eta_0 = \phi_0 + \tau_0.\\ &3.\quad h_0 = r_0 \sin\eta_0. \end{aligned}\right\}\ (1)$$

$$4.\quad \sin\eta_0 = \frac{h_0}{r_0}.$$

$$5.\quad \phi_0 = \eta_0 - \alpha_0.$$

$$6.\quad \sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$$

$$7.\quad \delta_0 = \phi_0 - \phi_0'.$$

$$8.\quad \alpha_0' = \delta_0 + \alpha_0.$$

$$9.\quad a_0' - r_0 = \frac{r_0 \sin\phi_0'}{\sin\alpha_0'}.$$

$$10.\quad \sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\alpha_0'}{r_2}.$$

$$11.\quad \sin\phi_2' = \sin\phi_2 \frac{n_{+1}}{n_{+3}}.$$

$$12.\quad \delta_2 = \phi_2 - \phi_2'.$$

$$13.\quad \alpha_2' = \delta_2 + \alpha_0.$$

$$14.\quad a_2' - r_2 = \frac{r_2 \sin\phi_2'}{\sin\alpha_2'}.$$

[*Continued on page 138*

The first three rays lie in one plane, which also contains the axis, while the fourth ray lies out of the axial plane.

In the first instance there will be given the formulæ for the first three rays, which involve a knowlege of the angle, τ_0, which the principal ray makes with the axis. The calculations also involve the distance ($e_{2,0}$) of the *first* principal point from the first refracting surface, a value which is obtained by reckoning through the lens in the reverse direction. The formulæ are as follows:—

II. When one Radius is large:— (2)

$$\left.\begin{array}{ll} 1. & \sin\phi_0 = \frac{(e_{2,0} - r_0)\sin\tau_0}{r_0}. \\ 2. & \eta_0 = \phi_0 + \tau_0. \\ 3. & h_0 = r_0 \sin\eta_0. \end{array}\right\} \text{(1)}$$

4. $$\sin\eta_0 = \frac{h_0}{r_0}.$$

5. $$\phi_0 = \eta_0 - \alpha_0.$$

6. $$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$$

7. $$\delta_0 = \phi_0 - \phi_0'.$$

8. $$\alpha_0' = \delta_0 + \alpha_0.$$

9. $$a_0' = \frac{h_0}{\tan\alpha_0'} + 2r_0 \sin^2\frac{\eta_0}{2}.$$

10. $$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\alpha_0'}{r_2}.$$

11. $$\sin\phi_2' = \sin\phi_2 \frac{n_{+1}}{n_{+3}}.$$

12. $$\delta_2 = \phi_2 - \phi_2'.$$

13. $$\alpha_2' = \delta_2 + \alpha_0'.$$

14. $$\eta_2 = \phi_2' + \alpha_2'.$$

[*Continued on page 139*

When no Radius is very large (*Continued*)

15. $h_2' = (a_2' - v_2') \tan \alpha_2'.$ (3)

$$\left.\begin{array}{ll} 16. & x = \dfrac{\sin \phi_0 \, r_0 \tan \alpha_0}{\sin \alpha_0}. \\[2ex] 17. & z^2 = x^2 + y^2. \\[2ex] 18. & \dfrac{x}{z} = \cos \zeta_0. \\[2ex] 19. & q_0 = z. \end{array}\right\} \text{(4)}$$

III. When the First Surface is plane:—

1. $h_0 = e_{2,0} \tan \tau_0.$ (1)

2. $\phi_0 = -\alpha_0.$

3. $\sin \phi_0' = \sin \phi_0 \dfrac{n_{-1}}{n_{+1}}.$

4. $\delta_0 = \phi_0 - \phi_0'.$

5. $\alpha_0' = \delta_0 + \alpha_0.$

6. $a_0' = \dfrac{h_0}{\tan \alpha_0'}.$

7. $\sin \phi_2 = \dfrac{(a_0' - r_2 - d_1) \sin \alpha_0'}{r_2}.$

8. $\sin \phi_2' = \sin \phi_2 \dfrac{n_{+1}}{n_{+3}}.$

9. $\delta_2 = \phi_2 - \phi_2'.$

10. $\alpha_2' = \delta_2 + \alpha_0'.$

11. $a_2' - r_2 = \dfrac{r_2 \sin \phi_2'}{\sin \alpha_2'}.$

12. $h_2' = (a_2' - v_2') \tan \alpha_2'.$ (3)

[*Continued on page 140*

When one Radius is large (*Continued*)

15. $h_2 = h_0 - d_1 \tan \alpha_0'.$

16. $a_2' = \dfrac{h_2}{\tan \alpha_2'} + 2r_2 \sin^2 \dfrac{\eta_2}{2}.$

17. $h_2' = (a_2' - v_2') \tan \alpha_2'.$ (3)

$$\left.\begin{aligned}
&\text{18.}\quad x = \frac{\sin \phi_0 \, r_0 \tan \alpha_0}{\sin \alpha_0}.\\
&\text{19.}\quad z^2 = x^2 + y^2.\\
&\text{20.}\quad \frac{x}{z} = \cos \zeta_0.\\
&\text{21.}\quad q_0 = z.
\end{aligned}\right\} \quad (4)$$

IV. When the Second Surface is plane:—

$$\left.\begin{aligned}
&\text{1.}\quad \sin \phi_0 = \frac{(e_{2,0} - r_0) \sin \tau_0}{r_0}.\\
&\text{2.}\quad \eta_0 = \phi_0 + \tau_0.\\
&\text{3.}\quad h_0 = r_0 \sin \eta_0.
\end{aligned}\right\} \quad (1)$$

4. $\sin \eta_0 = \dfrac{h_0}{r_0}.$

5. $\phi_0 = \eta_0 - \alpha_0.$

6. $\sin \phi_0' = \sin \phi_0 \dfrac{n_{-1}}{n_{+1}}.$

7. $\delta_0 = \phi_0 - \phi_0'.$

8. $\alpha_0' = \delta_0 + \alpha_0.$

9. $a_0' - r_0 = \dfrac{r_0 \sin \phi_0'}{\sin \alpha_0'}.$

10. $\phi_2 = -\alpha_0'.$

11. $\sin \phi_2' = \sin \phi_2 \dfrac{n_{+1}}{n_{+3}}.$

12. $\alpha_2' = -\phi_2'.$

[*Continued on page 141*

When the First Surface is plane (*Continued*)

$$\left.\begin{aligned} &13.[‡]\quad x = h_0.\\ &14.\quad z^2 = x^2 + y^2.\\ &15.\quad \frac{x}{z} = \cos \zeta_0.\\ &16.\quad q_0 = z. \end{aligned}\right\} \quad (4)$$

[‡] In this case the perpendicular plane of reference is the refracting surface itself.—Ed.

(1) These formulæ enable an approximate value of h_0 to be obtained for the principal ray. Using this value, the calculations can be made as far as the value of a_2', which is identical with $\tau_0 = a_0$ when the ray is a real principal ray; that is, when the inclination of the ray to the axis is the same on emergence as on entrance. By trial the correct value of h_0 for the principal ray can soon be obtained. The corresponding values for the upper and lower rim rays may then be found by means of the following formulæ, which can easily be derived:—

$$(104)\qquad \sin \phi_0 = \frac{\dfrac{h_0^\dagger}{\sin \tau_0} + e_{2,0} - r_0}{r_0} \sin \tau_0,$$

$$(105)\qquad \eta_0 = \phi_0 + \tau_0,$$

and

$$(106)\qquad h_0 = r_0 \sin \eta_0.$$

In these equations $h_0^\dagger$ is the radius of the incident beam of rays.[‡]

The formulæ for $\sin \eta_0$ are uniformly applicable to the cases of the principal ray and the upper and lower rim rays.

(2) The height formulæ may also be employed when it is desired to find the height at which a ray meets the lens surfaces.

(3) The term v_2' is the distance from the last refracting surface to the best focus plane.

[‡] It should be remembered that $h_0^\dagger$ is positive when measured above the principal ray, and negative when measured below.—Ed.

When the Second Surface is plane (*Continued*)

$$13.\quad a_2' = \frac{(a_0' - d_1)\tan\alpha_0'}{\tan\alpha_2'}.$$

$$14.\quad h_2 = a_2' \tan\alpha_2'.$$

$$15.\quad h_2' = (a_2' - v_2')\tan\alpha_2'. \quad (3)$$

$$\left.\begin{aligned} &16.\quad x = \frac{\sin\phi_0\, r_0 \tan\alpha_0}{\sin\alpha_0}. \\ &17.\quad s^2 = x^2 + y^2. \\ &18.\quad \frac{x}{s} = \cos\zeta_0. \\ &19.\quad q_0 = s. \end{aligned}\right\} (4)$$

---

(4) These formulæ serve for the determination of the constants which enter into the calculations for a ray that does not lie in a plane containing the axis. They are most conveniently annexed directly to the calculation of the rays in the axial plane, since the individual terms contained in the formulæ enter into this case, and are involved in the calculation of the principal ray.

The letter y represents the distance of the side ray from the axis.

Numerical Example of the Computation of Rays out of the Axis but in its Plane

As an example there will be chosen one of the lenses already discussed in this chapter, viz. the converging lens in which the deviations at the two faces are equal.

In the first instance the approximate value of the incidence height for the principal ray will be calculated. For this lens it was previously found that

$$e_{2,0} = +1{\cdot}281 \text{ mm.}‡$$

---

‡ With respect to the sign of $e_{2,0}$, it should be noted that in the previous computation, p. 116, of $e_{2,0}$ the value $-1{\cdot}281$ was obtained; that is, the position of $e_{2,0}$ with respect to the last refracting surface lay in the direction opposite to that of the incident light, and accordingly in the interior of the lens. If, as in the present case, the light is incident from the opposite side, it is then necessary to take $e_{2,0} = +1{\cdot}281$.

In addition it will be assumed that $\tau_0 = -6^\circ$. It follows that

$$\begin{aligned}
e_{2,0} &= 1{\cdot}281 \\
-r_0 &= \underline{-69{\cdot}250} \\
e_{2,0} - r_0 &= -67{\cdot}969 \\
\log (e_{2,0} - r_0) &= 1{\cdot}832\ 31n \\
\log (\sin \tau_0) &= 9{\cdot}019\ 23n \\
\log (1 : r_0) &= \underline{8{\cdot}159\ 58} \\
\log (\sin \phi_0) &= 9{\cdot}011\ 12 \\
\phi_0 &= 5^\circ\ 53'\ 18{\cdot}7'' \\
\tau_0 &= \underline{-6^\circ\ \ 0'\ \ 0''} \\
\eta_0 &= -0^\circ\ \ 6'\ 41{\cdot}3'' \\
\log (\sin \eta_0) &= 7{\cdot}288\ 74n \\
\log r_0 &= \underline{1{\cdot}840\ 42} \\
\log h_0 &= 9{\cdot}129\ 16\ n \\
h_0 &= -0{\cdot}134\ 64
\end{aligned}$$

Using this approximate value, $h_0 = -0{\cdot}134\ 64$, for the complete computation, it will be found that a_2' is not quite identical with $\tau_0 = a_0$. By trial and error h_0 may be altered until this identity is attained, when the exact value will be found to be $h_0 = -0{\cdot}134\ 35$ mm.

The incidence heights of the lower and upper rim rays are obtained as follows:—

Formulæ.	Lower Rim Ray.	Upper Rim Ray.
$\log h_0'$	1·176 09n	1·176 09
$\log (1 : \sin \tau_0)$	0·980 77n	0·980 77n
$\log (h_0' : \sin \tau_0)$	2·156 86	2·156 86n
$h_0' : \sin \tau_0$	143·502	− 143·502
$e_{2,0}$	1·281	1·281
$-r_0$	−69·250	− 69·250
$\frac{h_0'}{\sin \tau_0} + e_{2,0} - r_0$	75·533	− 211·471
$\log \left(\frac{h_0'}{\sin \tau_0} + e_{2,0} - r_0\right)$	1·878 14	2·325 25n
$\log \sin \tau_0$	9·019 23n	9·019 23n
$\log (1 : r_0)$	8·159 58	8·159 58
$\log \sin \phi_0$	9·056 95n	9·504 06
ϕ_0	− 6° 32′ 48″	18° 36′ 52″
τ_0	− 6 0 0	− 6 0 0
η_0	−12 32 48	12 36 52
$\log \sin \eta_0$	9·336 93n	9·339 23
$\log r_0$	1·840 42	1·840 42
$\log h_0$	1·177 35n	1·179 65
	mm.	mm.
h_0	−15·043 5	15·123 4

The further computation of the principal ray and the lower and upper rim rays for the D line is given in the following table. It should be remarked that v_2' in the case chosen (cf. p. 129) is equal to 94·180 mm., and that $y = 15$ mm.

Crown: ${}_{\mathrm{D}}n_{+1} = 1{\cdot}529\,64$ log ${}_{\mathrm{D}}n_{+1} = 0{\cdot}184\,59$

$\tau_0 = -6°$

$d_1 = 8{\cdot}0$ $r_0 = +\ 69{\cdot}250$ log $r_0 = 1{\cdot}840\,42$

$r_2 = -216{\cdot}195$ log $r_2 = 2{\cdot}334\,85\,n$

Formulæ.	Principal Ray.	Lower Rim Ray.	Upper Rim Ray.
h_0	−0·134 35	−15·043 5	+15·123 4
log h_0	9·128 24n	1·177 35n	1·179 65
log (1 : r_0)	8·159 58	8·159 58	8·159 58
log sin η_0	7·287 82n	9·336 93n	9·339 23
η_0	−0° 6′ 40·2″	−12° 32′ 48″	12° 36′ 52″
$-\tau_0 = -\alpha_0$	6 0 0	6	6
ϕ_0	5 53 19·8	− 6 32 48	18 36 52
log sin ϕ_0	9·011 14	9·056 95n	9·504 06
log (1 : n_{+1})	9·815 41	9·815 41	9·815 41
log sin ϕ_0'	8·826 55	8·872 36n	9·319 47
ϕ_0	5° 53′ 19·8″	− 6° 32′ 48″	18° 36′ 52″
$-\phi_0'$	−3 50 45·3	4 16 28	−12 2 41
δ_0	2 2 34·5	− 2 16 20	6 34 11
α_0	−6 0 0	− 6 0 0	− 6 0 0
α_0'	−3 57 25·5	− 8 16 20	0 34 11
log sin α_0'	8·838 91n	9·157 99n	7·997 53
log (1 : sin α_0')	1·161 09n	0·842 01n	2·002 47
log sin ϕ_0'	8·826 55	8·872 36n	9·319 47
log r_0	1·840 42	1·840 42	1·840 42
log ($a_0' - r_0$)	1·828 06n	1·554 79	3·162 36
$a_0' - r_0$	− 67·307	35·875	1 453·316
$r_0 - r_2 - d_1$	277·445	277·445	277·445
$a_0' - r_2 - d_1$	210·138	313·320	1 730·761
log ($a_0' - r_2 - d_1$)	2·322 51	2·495 99	3·238 24
log sin α_0'	8·838 91n	9·157 99n	7·997 53
log (1 : r_2)	7·665 15n	7·665 15n	7·665 15n
log sin ϕ_2	8·826 57	9·319 13	8·900 92n
log n_{+1}	0·184 59	0·184 59	0·184 59
log sin ϕ_2'	9·011 16	9·503 72	9·085 51n

Formulæ.	Principal Ray.	Lower Rim Ray.	Upper Rim Ray.
ϕ_2	3° 50′ 45·9″	12° 2′ 7″	− 4° 33′ 56·5″
$-\phi_2'$	−5 53 20·7	−18 35 57	6 59 37·5
δ_2	−2 2 34·8	− 6 33 50	2 25 41
α_0'	−3 57 25·5	− 8 16 20	0 34 11
α_2'	−6 0 0·3	−14 50 10	2 59 52
log sin α_2'	9·019 24*n*	9·408 33*n*	8·718 48
log (1 : sin α_2')	0·980 76*n*	0·591 67*n*	1·281 52
log sin ϕ_2'	9·011 16	9·503 72	9·085 51*n*
log r_2	2·334 85*n*	2·334 85*n*	2·334 85*n*
log $(a_2' - r_2)$	2·326 77	2·430 24	2·701 88
$a_2' - r_2$	212·212	269·302	503·362
$+r_2 - v_2'$	− 310·375	− 310·375	− 310·375
$a_2' - v_2'$	98·163	− 41·073	192·987
log $(a_2' - v_2')$	1·991 95*n*	1·613 56*n*	2·285 53
log tan α_2'	9·021 63*n*	9·423 06*n*	8·719 07
log h_2'	1·013 58	1·036 62	1·004 60
	mm.	mm.	mm.
h_2'	10·317 6	10·879 8	10·106 5
log sin ϕ_0	9·011 14		
log r_0	1·840 42		
log tan α_0	9·021 62		
log (1 : sin α_0)	0·980 77		
log x	0·853 95		
log x^2	1·707 90		
x^2	51·039		
y^2	225·0		
z^2	276·039		
log (z^2)	2·440 97		
log z	1·220 48		
log $(1 : z)$	8·779 52		
log x	0·853 95		
log cos ζ_0	9·633 47		
ζ_0	64° 31′ 56″		
$q_0 = z$	16·614		

Formulæ for the Computation of Rays lying out of the Axial Plane

Before considering the values obtained from the computation of the principal, lower, and upper rim rays, there will be computed an additional fourth ray proceeding from an object point lying out of the axis, viz. a side ray, situated at an equal distance from the upper and lower rim rays. This will be done in order to obtain complete information regarding the image surface corresponding with the chosen object point.

The values of ζ_0 and q_0 are obtained from the computation of the rays in the plane of the axis, and π_0 is to be taken as 0° or 180°, accordingly as the object point lying out of the axis is situated below or above the axis respectively.

The formulæ are as follows:—

I. When no Radius is Infinitely Great.	II. When the First Surface is Plane. ‖	III. When the Second Surface is Plane. ‖
$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$ †	$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$ †	$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$ †
$\sin\phi_0 = \frac{q_0\sin\lambda_0}{r_0}.$ ‡	$\phi_0 = \tau_0.$	$\sin\phi_0 = \frac{q_0\sin\lambda_0}{r_0}.$ ‡
$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\lambda_0' = \lambda_0+(\phi_0-\phi_0').$	$\lambda_0' = \lambda_0+(\phi_0-\phi_0').$	$\lambda_0' = \lambda_0+(\phi_0-\phi_0').$
$q_0' = \frac{r_0\sin\phi_0'}{\sin\lambda_0'} = q_0\frac{n_{-1}}{n_{+1}}\frac{\sin\lambda_0}{\sin\lambda_0'}.$ ‡	$q_0' = q_0.$	$q_0' = \frac{r_0\sin\phi_0'}{\sin\lambda_0'} = q_0\frac{n_{-1}}{n_{+1}}\frac{\sin\lambda_0}{\sin\lambda_0'}.$ ‡
$\sin\tau_0'\sin(\pi_0'-\zeta_0) = \frac{\sin\lambda_0'}{\sin\lambda_0}\sin\tau_0\sin(\pi_0-\zeta_0).$	$\tau_0' = \phi_0'.$	$\sin\tau_0'\sin(\pi_0'-\zeta_0) = \frac{\sin\lambda_0'}{\sin\lambda_0}\sin\tau_0\sin(\pi_0-\zeta_0).$
$\sin\tau_0'\cos(\pi_0'-\zeta_0) = \mp\cos\lambda_0'.$ †	$\pi_0' = \pi_0.$	$\sin\tau_0'\cos(\pi_0'-\zeta_0) = \mp\cos\lambda_0'.$ †
$\tan\tau_0' = \tan\tau_0\frac{\sin(\pi_0-\zeta_0)}{\sin(\pi_0'-\zeta_0)}.$ §	§	$\tan\tau_0' = \tan\tau_0\frac{\sin(\pi_0-\zeta_0)}{\sin(\pi_0'-\zeta_0)}.$ §
$\pm\frac{\sin(\phi_0-\phi_0')}{\sin(\pi_0-\pi_0')} = \frac{\sin\lambda_0\sin\tau_0'}{\sin(\pi_0-\zeta_0)} =$	$\sin\lambda_0\sin\tau_0' = \sin\lambda_0'\sin\tau_0.$	$\pm\frac{\sin(\phi_0-\phi_0')}{\sin(\pi_0-\pi_0')} = \frac{\sin\lambda_0\sin\tau_0'}{\sin(\pi_0-\zeta_0)} =$
$= \frac{\sin\lambda_0'\sin\tau_0}{\sin(\pi_0'-\zeta_0)}.$ †		$= \frac{\sin\lambda_0'\sin\tau_0}{\sin(\pi_0'-\zeta_0)}.$ †
$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$	$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$	$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$
$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$	$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$	$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$

† See p. 149.

‡ See p. 149.

§ These formulæ serve as checks.

‖ See p. 149.

[*Continued over.*

I. When no Radius is Infinitely Great.	II. When the First Surface is Plane. ‖	III. When the Second Surface is Plane. ‖
$\tau_2 = \tau_0'.$	$\tau_2 = \tau_0'.$	$\tau_2 = \tau_0'.$
$\pi_2 = \pi_0'.$	$\pi_2 = \pi_0'.$	$\pi_2 = \pi_0'.$
$c_1 = d_1 - r_0 + r_2.$	$c_1 = d_1 + r_2.$	$c_1 = d_1 - r_0.$
$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$	$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$	$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$
$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) - c_1 \tan\tau_0'.$	$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) - c_1 \tan\tau_0'.$	$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) - c_1 \tan\tau_0'.$
$\frac{c_1 \tan\tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.$ §	$\frac{c_1 \tan\tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.$ §	$\frac{c_1 \tan\tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.$ §
$\cos\lambda_2 = \mp\sin\tau_2 \cos(\pi_2 - \zeta_2).$ †	$\cos\lambda_2 = \mp\sin\tau_2 \cos(\pi_2 - \zeta_2).$ †	$\cos\lambda_2 = \mp\sin\tau_2 \cos(\pi_2 - \zeta_2).$ †
$\sin\phi_2 = \frac{q_2 \sin\lambda_2}{r_2}.$ ‡	$\sin\phi_2 = \frac{q_2 \sin\lambda_2}{r_2}.$ ‡	$\phi_2 = \tau_2.$
$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$
$\lambda_2' = \lambda_2 + (\phi_2 - \phi_2').$	$\lambda_2' = \lambda_2 + (\phi_2 - \phi_2').$	$\lambda_2' = \lambda_2 + (\phi_2 - \phi_2').$
$q_2' = \frac{r_2 \sin\phi_2'}{\sin\lambda_2'} = q_2 \frac{n_1}{n_3} \frac{\sin\lambda_2}{\sin\lambda_2'}.$ ‡	$q_2' = \frac{r_2 \sin\phi_2'}{\sin\lambda_2'} = q_2 \frac{n_1}{n_3} \frac{\sin\lambda_2}{\sin\lambda_2'}.$ ‡	$q_2' = q_2.$
$\sin\tau_2' \sin(\pi_2' - \zeta_2) = \frac{\sin\lambda_2'}{\sin\lambda_2} \sin\tau_2 \sin(\pi_2 - \zeta_2).$	$\sin\tau_2' \sin(\pi_2' - \zeta_2) = \frac{\sin\lambda_2'}{\sin\lambda_2} \sin\tau_2 \sin(\pi_2 - \zeta_2).$	$\tau_2' = \phi_2'.$

† See p. 149.

‡ See p. 149.

§ These formulæ serve as checks.

‖ See p. 149.

$\sin\tau_2'\cos(\pi_2'-\zeta_2) = \mp\cos\lambda_2'.\dagger$	$\sin\tau_2'\cos(\pi_2'-\zeta_2) = \mp\cos\lambda_2'.\dagger$	$\pi_2 = \pi_2'.$
$\left\{\begin{aligned} &\tan\tau_2' = \tan\tau_2\frac{\sin(\pi_2-\zeta_2)}{\sin(\pi_2'-\zeta_2)}. \\ &\pm\frac{\sin(\phi_2-\phi_2')}{\sin(\pi_2-\pi_2')} = \frac{\sin\lambda_2\sin\tau_2'}{\sin(\pi_2-\zeta_2)} = \\ &\qquad = \frac{\sin\lambda_2'\sin\tau_2}{\sin(\pi_2'-\zeta_2)}.\dagger \\ &\sin\epsilon_2 = \sin\tau_2'\sin\pi_2'. \\ &\frac{\sin(\phi_2-\phi_2')\sin(\phi_2+\phi_2')}{\sin\phi_2\sin\phi_2'} = 2\cot 2\omega. \end{aligned}\right\}\S$	$\left\{\begin{aligned} &\tan\tau_2' = \tan\tau_2\frac{\sin(\pi_2-\zeta_2)}{\sin(\pi_2'-\zeta_2)}. \\ &\pm\frac{\sin(\phi_2-\phi_2')}{\sin(\pi_2-\pi_2')} = \frac{\sin\lambda_2\sin\tau_2'}{\sin(\pi_2-\zeta_2)} = \\ &\qquad = \frac{\sin\lambda_2'\sin\tau_2}{\sin(\pi_2'-\zeta_2)}.\dagger \\ &\sin\epsilon_2 = \sin\tau_2'\sin\pi_2'. \\ &\frac{\sin(\phi_2-\phi_2')\sin(\phi_2+\phi_2')}{\sin\phi_2\sin\phi_2'} = 2\cot 2\omega. \end{aligned}\right\}\S$	$\left\{\begin{aligned} &\sin\lambda_2\sin\tau_2' = \sin\lambda_2'\sin\tau_2. \\ &\sin\epsilon_2 = \sin\tau_2'\sin\pi_2'. \\ &\frac{\sin(\phi_2-\phi_2')\sin(\phi_2+\phi_2')}{\sin\phi_2\sin\phi_2'} = 2\cot 2\omega. \end{aligned}\right\}\S$
$c_3 = b_2' - r_2.\P$	$c_3 = b_2' - r_2.\P$	$c_3 = b_2'.\P$
$z\sin(\pi_2'-\xi) = q_2'\sin(\pi_2'-\zeta_2).\P$	$z\sin(\pi_2'-\xi) = q_2'\sin(\pi_2'-\zeta_2).\P$	$z\sin(\pi_2'-\xi) = q_2'\sin(\pi_2'-\zeta_2).\P$
$z\cos(\pi_2'-\xi) = q_2'\cos(\pi_2'-\zeta_2) - c_3\tan\tau_2'.\P$	$z\cos(\pi_2'-\xi) = q_2'\cos(\pi_2'-\zeta_2) - c_3\tan\tau_2'.\P$	$z\cos(\pi_2'-\xi) = q_2'\cos(\pi_2'-\zeta_2) - c_3\tan\tau_2'.\P$
$\frac{c_3\tan\tau_2'}{\sin(\zeta_2-\xi)} = \frac{z}{\sin(\pi_2'-\zeta_2)} = \frac{q_2'}{\sin(\pi_2'-\xi)}.\ \P\,\&\,\S$	$\frac{c_3\tan\tau_2'}{\sin(\zeta_2-\xi)} = \frac{z}{\sin(\pi_2'-\zeta_2)} = \frac{q_2'}{\sin(\pi_2'-\xi)}.\ \P\,\&\,\S$	$\frac{c_3\tan\tau_2'}{\sin(\zeta_2-\xi)} = \frac{z}{\sin(\pi_2'-\zeta_2)} = \frac{q_2'}{\sin(\pi_2'-\xi)}.\ \P\,\&\,\S$

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§ These formulæ serve as checks.

‖ In the case of a plane surface whose centre is at infinity, the plane of the surface itself is substituted for the centre plane. The points P, Q, and Q′ then coincide, and their co-ordinates and radii vectors are identical. The interval *c* is measured from the surface to the appropriate centre plane; it is no longer the separation of two consecutive centre planes.—Ed.

¶ These formulæ refer to a final plane which is not necessarily a centre plane. The interval *c* then represents the separation of the final plane from the previous centre plane The polar co-ordinates of the point of intersection of the emergent ray with the final plane are *z* and *ξ*. See fig. 8.—Ed.

† Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—Ed.

‡ In these cases *r* is always positive whether the surface is convex or concave.—Ed.

Example of the Computation of Rays lying out of the Axial Plane

For the following example of the computation of a side ray for the colour D there has been chosen the lens which is the basis of the example discussed in the previous section, p. 141.

Crown: ${}_{\mathrm{D}}n_{+1} = 1{\cdot}529\,64 \quad \log {}_{\mathrm{D}}n_{+1} = 0{\cdot}184\,59$

$\tau_0 = 6°;\ \zeta_0 = 64°\ 31'\ 56'';\ \pi_0 = 180°$

$\pi_0 - \zeta_0 = 115°\ 28'\ 4'';\ q_0 = 16{\cdot}614$

$r_0 = +\ 69{\cdot}250 \quad \log r_0 = 1{\cdot}840\,42$

$r_2 = -216{\cdot}195 \quad \log r_2 = 2{\cdot}334\,85n$

$d_1 = 8$

$\log(\mp \sin \tau_0)$ †	$9{\cdot}019\,23n$	$\log \sin(\phi_0 + \phi_0')$	$9{\cdot}589\,76$
$\log \cos(\pi_0 - \zeta_0)$	$9{\cdot}633\,47n$	$\log \sin(\phi_0 - \phi_0')$	$8{\cdot}927\,32$
$\log \cos \lambda_0$	$8{\cdot}652\,70$	$\log 1 : \sin \phi_0$	$0{\cdot}620\,38$
		$\log 1 : \sin \phi_0'$	$0{\cdot}804\,97$
λ_0	$87°\ 25'\ 26''$	$\log 2 \cot 2\omega$	$9{\cdot}942\,43$
$\log \sin \lambda_0$	$9{\cdot}999\,56$	$\log r_0$	$1{\cdot}840\,42$
$\log 1 : r_0$ ‡	$8{\cdot}159\,58$	$\log \sin \phi_0'$	$9{\cdot}195\,03$
$\log q_0$	$1{\cdot}220\,48$	$\log 1 : \sin \lambda_0'$	$0{\cdot}000\,34$
$\log \sin \phi_0$	$9{\cdot}379\,62$	$\log q_0'$	$1{\cdot}035\,79$
$\log n_{-1} : n_{+1}$	$9{\cdot}815\,41$		
$\log \sin \phi_0'$	$9{\cdot}195\,03$	$\log \sin \lambda_0 : \sin \lambda_0'$	$9{\cdot}999\,90$
ϕ_0	$13°\ 52'\ 2''$	$\log n_{-1} : n_{+1}$	$9{\cdot}815\,41$
ϕ_0'	$9\ \ 0\ \ 53$	$\log q_0$	$1{\cdot}220\,48$
$\phi_0 - \phi_0'$	$4\ \ 51\ \ 9$	$\log q_0'$	$1{\cdot}035\,79$
λ_0	$87\ \ 25\ \ 26$		
λ_0'	$92\ \ 16\ \ 35$	$\log \sin \lambda_0' : \sin \lambda_0$	$0{\cdot}000\,10$
		$\log \sin \tau_0$	$9{\cdot}019\,23$
$\log \sin \lambda_0'$	$9{\cdot}999\,66$	$\log \sin(\pi_0 - \zeta_0)$	$9{\cdot}955\,61$
$\log \cos \lambda_0'$	$8{\cdot}599\,01n$	$\log \sin \tau_0' \sin(\pi_0' - \zeta_0)$	$8{\cdot}974\,94$
		$-\log(\mp \cos \lambda_0')$ †	$8{\cdot}599\,01$
$\phi_0 + \phi_0'$	$22°\ 52'\ 55''$	$\log \tan(\pi_0' - \zeta_0)$	$0{\cdot}375\,93$

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† See note, p. 149.

‡ See note, p. 149.

$(\pi_0' - \zeta_0)$	67° 10′ 44″
ζ_0	64 31 56
π_0'	131 42 40
π_0	180 0 0
$\pi_0 - \pi_0'$	48 17 20
$\log \sin (\pi_0' - \zeta_0)$	9·964 60
$\log \cos (\pi_0' - \zeta_0)$	9·588 67
$\log \sin \tau_0' \sin (\pi_0' - \zeta_0)$	8·974 94
$\log 1 : \sin (\pi_0' - \zeta_0)$	0·035 40
$\log \sin \tau_0'$	9·010 34
$\log \tan \tau_0$	9·021 62
$\log \sin (\pi_0 - \zeta_0)$	9·955 61
$\log 1 : \sin (\pi_0' - \zeta_0)$	0·035 40
$\log \tan \tau_0'$	9·012 63
$\log \sin \lambda_0$	9·999 56
$\log \sin \tau_0'$	9·010 34
$\log 1 : \sin (\pi_0 - \zeta_0)$	0·044 39
	9·054 29
$\log \sin \lambda_0'$	9·999 66
$\log \sin \tau_0$	9·019 23
$\log 1 : \sin (\pi_0' - \zeta_0)$	0·035 40
	9·054 29
$\log \pm \sin (\phi_0 - \phi_0')$ †	8·927 32
$-\log \sin (\pi_0 - \pi_0')$	9·873 04
	9·054 28
$\log \sin \tau_0'$	9·010 34
$\log \sin \pi_0'$	9·873 04
$\log \sin \epsilon_0$	8·883 38
$c_1 = d_1 - r_0 + r_2$ ‡	−277·445
$\log q_0'$	1·035 79
$\log \cos (\pi_0' - \zeta_0)$	9·588 67
$\log q_0' \cos (\pi_0' - \zeta_0)$	0·624 46

$\log (-c_1)$	2·443 18
$\log \tan \tau_0'$	9·012 63
$\log (-c_1 \tan \tau_0')$	1·455 81
$q_0' \cos (\pi_0' - \zeta_0)$	4·211 73
$-c_1 \tan \tau_0'$	28·563 4
$q_2 \cos (\pi_0' - \zeta_2)$	32·775 1
$\log q_0'$	1·035 79
$\log \sin (\pi_0' - \zeta_0)$	9·964 60
$\log q_2 \sin (\pi_0' - \zeta_2)$	1·000 39
$-\log q_2 \cos (\pi_0' - \zeta_2)$	1·515 54
$\log \tan (\pi_0' - \zeta_2)$	9·484 85
$\pi_0' - \zeta_2$	16° 58′ 55″
π_0'	131 42 40
ζ_2	114 43 45
ζ_0	64 31 56
$\zeta_0 - \zeta_2$	−50 11 49
$\log \sin (\pi_0' - \zeta_2)$	9·465 49
$\log \cos (\pi_0' - \zeta_2)$	9·980 64
$\log q_2 \sin (\pi_0' - \zeta_2)$	1·000 39
$\log 1 : \sin (\pi_0' - \zeta_2)$	0·534 51
$\log q_2$	1·534 90
$\log 1 : \sin (\pi_0' - \zeta_0)$	0·035 40
	1·570 30
$\log q_0'$	1·035 79
$\log 1 : \sin (\pi_0' - \zeta_2)$	0·534 51
	1·570 30
$\log c_1 \tan \tau_0'$	1·455 81*n*
$\log 1 : \sin (\zeta_0 - \zeta_2)$	0·114 50*n*
	1·570 31
$\log (\mp \sin \tau_2)$ †	9·010 34
$\log \cos (\pi_2 - \zeta_2)$	9·980 64
$\log \cos \lambda_2$	8·990 98
λ_2	84° 22′ 45″

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† See note, p. 149.

‡ See note, p. 149.

Crown: ${}_{D}n_{+1} = 1{\cdot}529\,64$ $\log {}_{D}n_{+1} = 0{\cdot}184\,59$
$\tau_0 = 6°$; $\zeta_0 = 64° \; 31' \; 56''$; $\pi_0 = 180°$
$\pi_0 - \zeta_0 = 115° \; 28' \; 4''$; $q_0 = 16{\cdot}614$
$r_0 = +\,69{\cdot}250$ $\log r_0 = 1{\cdot}840\,42$
$r_2 = -216{\cdot}195$ $\log r_2 = 2{\cdot}334\,85n$
$d_1 = 8$

$\log \sin \lambda_2$	$9{\cdot}997\,91$	$\log \sin \lambda_2' : \sin \lambda_2$	$9{\cdot}994\,75$
$\log 1 : r_2$ ‡	$7{\cdot}665\,15$	$\log \sin \tau_2$	$9{\cdot}010\,34$
$\log q_2$	$1{\cdot}534\,90$	$\log \sin (\pi_0' - \zeta_2)$	$9{\cdot}465\,49$
$\log \sin \phi_2$	$9{\cdot}197\,96$	$\log \sin \tau_2' \sin (\pi_2' - \zeta_2)$	$8{\cdot}470\,58$
$\log n_1 : n_3$	$0{\cdot}184\,59$	$-\log (\mp \cos \lambda_2')$ †	$9{\cdot}260\,95$
$\log \sin \phi_2'$	$9{\cdot}382\,55$	$\log \tan (\pi_2' - \zeta_2)$	$9{\cdot}209\,63$
ϕ_2	$9° \; 4' \; 34''$	$\pi_2' - \zeta_2$	$9° \; 12' \; 16''$
ϕ_2'	$13 \; 57 \; 47$	ζ_2	$114 \; 43 \; 45$
$\phi_2 - \phi_2'$	$-4 \; 53 \; 13$	π_2'	$123 \; 56 \; 1$
λ_2	$84 \; 22 \; 45$	π_2	$131 \; 42 \; 40$
λ_2'	$79 \; 29 \; 32$	$\pi_2 - \pi_2'$	$7 \; 46 \; 39$
		$\log \sin (\pi_2' - \zeta_2)$	$9{\cdot}204\,01$
$\log \sin \lambda_2'$	$9{\cdot}992\,66$	$\log \cos (\pi_2' - \zeta_2)$	$9{\cdot}994\,37$
$\log \cos \lambda_2'$	$9{\cdot}260\,95$		
		$\log \sin \tau_2' \sin (\pi_2' - \zeta_2)$	$8{\cdot}470\,58$
$\phi_2 + \phi_2'$	$23° \; 2' \; 21''$	$\log 1 : \sin (\pi_2' - \zeta_2)$	$0{\cdot}795\,99$
$\log \sin (\phi_2 + \phi_2')$	$9{\cdot}592\,58$	$\log \sin \tau_2'$	$9{\cdot}266\,57$
$\log \sin (\phi_2 - \phi_2')$	$8{\cdot}930\,39$		
$\log 1 : \sin \phi_2$	$0{\cdot}802\,04$	$\log \tan \tau_2$	$9{\cdot}012\,63$
$\log 1 : \sin \phi_2'$	$0{\cdot}617\,45$	$\log \sin (\pi_0' - \zeta_2)$	$9{\cdot}465\,49$
$\log 2 \cot 2\omega$	$9{\cdot}942\,46$	$\log 1 : \sin (\pi_2' - \zeta_2)$	$0{\cdot}795\,99$
		$\log \tan \tau_2'$	$9{\cdot}274\,11$
$\log r_2$ ‡	$2{\cdot}334\,85$		
$\log \sin \phi_2'$	$9{\cdot}382\,55$	$\log \sin \lambda_2$	$9{\cdot}997\,91$
$\log 1 : \sin \lambda_2'$	$0{\cdot}007\,34$	$\log \sin \tau_2'$	$9{\cdot}266\,57$
$\log q_2'$	$1{\cdot}724\,74$	$\log 1 : \sin (\pi_2 - \zeta_2)$	$0{\cdot}534\,51$
			$9{\cdot}798\,99$
$\log \sin \lambda_2 : \sin \lambda_2'$	$0{\cdot}005\,25$	$\log \sin \lambda_2'$	$9{\cdot}992\,66$
$\log n_1 : n_3$	$0{\cdot}184\,59$	$\log \sin \tau_2$	$9{\cdot}010\,34$
$\log q_2$	$1{\cdot}534\,90$	$\log 1 : \sin (\pi_2' - \zeta_2)$	$0{\cdot}795\,99$
$\log q_2'$	$1{\cdot}724\,74$		$9{\cdot}798\,99$

† See note, p. 149. ‡ See note, p. 149.

$\log \pm \sin (\phi_2 - \phi_2')$ †	8·930 39	$\pi_2' - \xi$	126° 11′ 29″
$-\log \sin (\pi_2 - \pi_2')$	9·131 38	π_2'	123 56 1
	9·799 01	$-\xi$	2 15 28
		ζ_2	114 43 45
$\log \sin \tau_2'$	9·266 57	$\zeta_2 - \xi$	116 59 13
$\log \sin \pi_2'$	9·918 91		
$\log \sin \epsilon_2$	9·185 48		
		$\log \sin (\pi_2' - \xi)$	9·906 90
$c_3 = b_2' - r_2$ ‡	311·648	$\log \cos (\pi_2' - \xi)$	9·771 21
$\log q_2'$	1·724 74		
		$\log z \sin (\pi_2' - \xi)$	0·928 75
$\log \cos (\pi_2' - \zeta_2)$	9·994 37		
		$\log 1 : \sin (\pi_2' - \xi)$	0·093 10
$\log q_2' \cos (\pi_2' - \zeta_2)$	1·719 11		
		$\log z$	1·021 85
$\log (-c_3)$	2·493 66*n*	$\log 1 : \sin (\pi_2' - \zeta_2)$	0·795 99
$\log \tan \tau_2'$	9·274 11		1·817 84
$\log (-c_3 \tan \tau_2')$	1·767 77*n*		
		$\log q_2'$	1·724 74
$q_2' \cos (\pi_2' - \zeta_2)$	52·373 3	$\log 1 : \sin (\pi_2' - \xi)$	0·093 10
$-c_3 \tan \tau_2'$	−58·582 8		
			1·817 84
$z \cos (\pi_2' - \xi)$	− 6·209 5		
$\log q_2'$	1·724 74	$\log c_3 \tan \tau_2'$	1·767 77*n*
$\log \sin (\pi_2' - \zeta_2)$	9·204 01	$\log 1 : \sin (\zeta_2 - \xi)$	0·050 07*n*
$\log z \sin (\pi_2' - \xi)$	0·928 75		1·817 84
$-\log z \cos (\pi_2' - \xi)$	0·793 06*n*		
$\log \tan (\pi_2' - \xi)$	0·135 69*n*	z	10·5160

† See note, p. 149.

‡ See note, p. 149.

Aberrations of Rays proceeding from an Object Point lying out of the Axis

In Chapter III, p. 56, it has been stated that in order that the image corresponding with an object lying out of the axis may be clearly defined and undistorted, it is necessary that the three chosen rays of the beam, namely, the upper, the lower, and the side rays, should intersect one another on the principal ray, and that the distance of this point of intersection from the principal point should be equal to the true focal length. Further, the principal ray should so emerge as if it proceeded from the second principal point.

There will now be considered what are the errors that arise in a particular lens, and in the first instance the results of computation of the principal ray and the upper, lower, and side rim rays in the case of the example already selected on several occasions, namely, that in which the refraction at the two lens surfaces is the same, have been summarized in Tables IX and X, p. 155:—

In Table IX there are indicated for the principal ray and the upper and lower rim rays their points of intersection with the plane normal to the axis at a distance of 95·453 mm. behind the vertex of the last refracting surface, or 311·648 mm. from the last centre plane, that is, the normal plane containing the centre of the last lens surface. The position of the point is indicated by its distance h_2' from the axis, and the angle a_2' which the emergent ray makes with the axis. The results are given for both D and F rays. The chosen plane normal to the axis in which the image point should lie is distant from the principal point by the following amount:—

$$100 \times \cos 6^\circ = 99{\cdot}452 \text{ mm.}$$

Therefore its distance behind the last refracting surface is

$$b_2' = 99{\cdot}452 - 3{\cdot}999 = 95{\cdot}453 \text{ mm.}$$

Results similar to those for the rim rays are given in the table for the $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, and $\frac{16}{81}$ rim rays.

IX. OBLIQUE RAYS IN THE AXIAL PLANE

For $b_2' = 95{\cdot}453$ mm.	D.		F.	
h_0.	h_2'.	α_2'.	h_2'.	α_2'.
mm.	mm.		mm.	
15·0	10·040	+ 2° 59′ 52″	9·846	+ 3° 6′ 40″
10·0	10·274	− 0 4 9	10·144	− 0 0 19
6·666 7	10·354	− 2 3 52	10·268	− 2 0 54
4·444 4	10·391	− 3 23 4	10·333	− 3 20 53
2·963 0	10·412	− 4 15 12	10·374	− 4 14 3
0·0	10·452	− 6 0 0	10·453	− 6 0 18
− 2·963 0	10·499	− 7 42 13	10·534	− 7 43 32
− 4·444 4	10·532	− 8 33 31	10·588	− 8 35 30
− 6·666 7	10·602	− 9 50 57	10·687	− 9 53 56
−10·0	10·767	−11 48 35	10·900	−11 53 5
−15·0	11·217	−14 50 10	11·426	−14 57 3

X. RAYS NOT IN THE AXIAL PLANE

For $b_2' = 95{\cdot}453$ mm.	D.				F.			
h_0.	z.	ξ.	π_2'.	τ_2'.	z.	ξ.	π_2'.	τ_2'.
	mm.				mm.			
15	10·516	−2° 15′ 28″	123° 56′ 1″	10° 38′ 46″	10·527	−3° 21′ 9″	123° 35′ 44″	10° 44′ 27″
10	10·477	−0 45 25	135 46 15	8 20 17	10·477	−1 27 1	135 24 45	8 23 25
6·666 7	10·463	−0 17 27	147 13 20	7 7 24	10·462	−0 45 16	146 53 39	7 9 7
4·444 4	10·456	−0 7 42·5	156 50 22	6 31 15	10·455	−0 27 8·5	156 34 43	6 32 2
2·963 0	10·453	−0 4 4·2	164 6 15	6 14 10	10·453	−0 17 7	163 20 20	6 17 33

In Table X, which refers to the side ray, there are indicated the values of

- the angle τ_2' which the emergent ray makes with a ray drawn parallel to the axis;
- the angle π_2' which the projection of the emergent ray on the above-mentioned image plane normal to the axis makes with the vertical drawn upwards from its point of intersection with the plane;
- the radius vector z in the plane in question, i.e. the distance from the axis to the point of intersection of the emergent ray with the plane; and finally,
- the angle ξ which the radius vector z makes with a vertical drawn upwards.

If the rectangular co-ordinates of the point of intersection of the emergent ray with the image plane are x and y, then

$$(104) \qquad x = z \cos \xi \quad \text{and} \quad y = z \sin \xi.$$

The results thus obtained are indicated in Table XI.

XI. RAYS NOT IN THE AXIAL PLANE

For $b_2' = 95{\cdot}453$ mm.	D.		F.	
h_0.	x.	y.	x.	y.
15	10·508	−0·414	10·509	−0·616
10	10·476	−0·138	10·474	−0·265
6·667	10·462	−0·053	10·464	−0·138
4·444	10·456	−0·023	10·455	−0·080
2·963	10·453	−0·012	10·454	−0·052

Before discussing the last two tables there will be calculated, for an image point corresponding with an object point lying out of the axis in the case of an *ideally perfect* lens, the height h_2' above the axis and the angle of inclination a_2' of the emergent oblique ray that passes through the image point and lies in the plane of the axis. The results are summarized in Table XII, while Table XIII gives the values of z, ξ, π_2', and τ_2' for the ray which passes through the ideal lens to the image point, and lies out of the plane of the axis.

XII. OBLIQUE RAYS IN THE AXIAL PLANE (IDEAL LENS)

For $b_2' = 95{\cdot}453$ mm., alike for D and F.		
h_0.	h_2'.	α_2'.
	mm.	
15·0	10·452	+ 2° 37′ 48″
10·0	10·452	− 0 15 36
6·666 7	10·452	− 2 10 37
4·444 4	10·452	− 3 27 9
2·963 0	10·452	− 4 18 5
0	10·452	− 6 0 0
− 2·963 0	10·452	− 7 41 52
− 4·444 4	10·452	− 8 32 48
− 6·666 7	10·452	− 9 49 21
− 10·0	10·452	− 11 44 23
− 15·0	10·452	− 14 37 47

XIII. RAYS NOT IN THE AXIAL PLANE (IDEAL LENS)

For $b_2' = 95{\cdot}453$ mm., alike for D and F.				
h_0.	z.	ξ.	π_2'.	τ_2'.
15·0	10·452	0° 0′ 0″	124° 52′ 38″	10° 31′ 57″
10·0	10·452	0 0 0	136 16 6	8 19 2
6·666 7	10·452	0 0 0	147 27 15	7 7 23
4·444 4	10·452	0 0 0	156 57 35	6 31 19
2·963 0	10·452	0 0 0	164 10 27	6 14 14
0·0	10·452	0 0 0	180 0 0	6 0 0

The figures contained in the tables can be used not only to determine the ideal values for a perfect lens, but also to determine in the case of the actual lens under consideration the values of the distances to the points of intersection with the principal ray measured from the principal point, as well as the angular apertures measured from the principal ray, and finally the positions of the projected virtual emergence points on the principal ray. These results are indicated in Table XIV in Groups 1, 2, and 3, while the errors deduced from them are summarized under Groups 4, 5, and 6.

XIV. ERRORS IN THE FOCAL LENGTH FOR AN IMAGE POINT SITUATED OUT OF THE AXIS

		+15·0	+10·0	+6·666 7	+4·444 4	+2·963 0	−2·963 0	−4·444 4	−6·666 7	−10·0	−15·0
1. Intersections with principal ray measured from principal point:		mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
(*a*) Ideal values:		100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0	100·0
(*b*) Actual values:	D	97·382	98·270	98·574	98·665	98·680	98·421	98·210	97·769	96·892	95·027
	F	96·173	97·056	97·352	97·428	97·469	97·276	96·992	96·544	95·630	93·740
2. Angular apertures measured from principal ray:											
(*a*) Ideal values:		30 983″	20 674″	13 789″	9 200″	6 137″	6 155″	9 242″	13 886″	20 887″	31 484″
(*b*) Actual values:	D	32 392	21 351	14 168	9 416	6 288	6 133	9 211	13 857	20 915	31 810
	F	32 800	21 581	14 346	9 547	6 357	6 212	9 330	14 036	21 185	32 223
3. Virtual emergence points projected on principal ray:		mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
(*a*) Ideal values:		+2·681	+1·545	+0·924	+0·566	+0·356	−0·270	−0·355	−0·475	−0·550	−0·432
(*b*) Actual values:	D	+2·296	+1·333	+0·831	+0·526	+0·345	−0·279	−0·409	−0·572	−0·769	−0·956
	F	+2·209	+1·350	+0·845	+0·544	+0·358	−0·266	−0·394	−0·554	−0·753	−0·943
4. Errors in the position of the intersection points:	D	2·618	1·720	1·426	1·335	1·310	1·579	1·790	2·231	3·108	4·973
	F	3·827	2·944	2·648	2·572	2·531	2·723	3·008	3·456	4·370	6·260
5. Errors in the angular aperture:	D	−1 409″	−677″	−379″	−216″	−149″	+22″	+31″	+ 29″	− 28″	−326″
	F	−1 817	−907	−557	−347	−220	−57	−88	−150	−298	−739
6. Errors in the virtual emergence points		mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
	D	+0·584	+0·213	+0·088	+0·041	+0·012	+0·012	+0·042	+0·095	+0·222	+0·523
	F	+0·474	+0·194	+0·076	+0·021	−0·004	−0·004	+0·027	+0·076	+0·206	+0·511

XV. COLOUR CORRESPONDING TO LINE D

	15 mm.		10 mm.		6·667 mm.		4·444 mm.		2·963 mm.		Principal Ray.
	x	y	x	y	x	y	x	y	x	y	
Upper rim ray	10·040		10·274		10·354		10·391		10·412		10·452
Lower rim ray	11·217		10·767		10·602		10·532		10·499		
Side rim ray	10·508	−0·414	10·476	−0·138	10·462	−0·053	10·456	−0·023	10·453	−0·012	

COLOUR CORRESPONDING TO LINE F

	15 mm.		10 mm.		6·667 mm.		4·444 mm.		2·963 mm.		Principal Ray.
	x	y	x	y	x	y	x	y	x	y	
Upper rim ray	9·846		10·144		10·268		10·333		10·374		10·453
Lower rim ray	11·426		10·900		10·687		10·588		10·534		
Side rim ray	10·509	−0·616	10·474	−0·265	10·464	−0·138	10·455	−0·080	10·454	−0·052	

XVI. COLOUR CORRESPONDING TO LINE D

	15 mm.		10 mm.		6·667 mm.		4·444 mm.		2·963 mm.		Principal Ray.
	x	y	x	y	x	y	x	y	x	y	
Upper rim ray	10·250		10·269		10·210		10·155		10·116		10·032
Lower rim ray	10·158		9·931		9·908		9·930		9·958		
Side rim ray	10·193	0·053	10·161	0·168	10·147	0·150	10·141	0·111	10·138	0·077	

COLOUR CORRESPONDING TO LINE F

	15 mm.		10 mm.		6·667 mm.		4·444 mm.		2·963 mm.		Principal Ray.
	x	y	x	y	x	y	x	y	x	y	
Upper rim ray	10·063		10·144		10·127		10·099		10·077		10·032
Lower rim ray	10·358		10·058		9·989		9·984		9·991		
Side rim ray	10·194	−0·142	10·159	0·045	10·146	0·068	10·139	0·057	10·137	0·043	

A full discussion of the linear and angular errors contained in Table XIV may be dispensed with, since the results cannot be easily expressed in a clear form. It will be seen that even in the lens of best form, in which the deviations at the two surfaces are equal, the aberrations of the image produced by an incident oblique beam are very considerable, and, since all the errors are simultaneously present, they cannot be represented in a simple manner. For example, it is not possible to specify an image surface, since an image point falls far short of representing an object point.

A true idea of the image produced by a lens can best be obtained from diagrams. On Plate I, fig. 55, there are indicated the positions in the vertical axial plane of the principal and the upper and lower rim rays of an incident beam of light for semi-diameters of 15 mm., 10 mm., 6·667 mm., 4·444 mm., and 2·963 mm. Rays of colour D are shown red, and those for colour F blue. In the diagram there is also indicated the line of intersection of the vertical plane containing the axis with the chosen image plane (at a distance of 311·648 mm. from the centre of the last surface), and the line of intersection with the same axial plane of a second vertical image plane at a distance of 308·648 mm. from the centre of the last surface. It will be seen at once that in this latter plane the spread of the rays is very much smaller. The diffused image will accordingly be smaller provided the side rays are not appreciably worse, which, however, is not the case, as may be confirmed by calculating the intersection points for a side ray.

Fig. 56, Plate II, shows the diffused image in the plane whose distance from the last centre plane is 311·648 mm. The image is magnified 200-fold. The values from which the diagram is made are summarized in the small table (XV).

From these numbers it will be seen at once that all the rays have crossed one another, since the upper rim rays meet the image plane below the lower rim rays, and since also the side rays intersect on the side of the principal ray, opposite that of their original position. In the diagram, those rays which originally lay on the surface of a cylindrical beam are joined by curves. Further, there is also indicated a diagram, fig. 57, Plate II, magnified two-fold, of a section of the incident beam by a plane normal to the lens axis. The chosen cylinders are indicated alternately by full and dotted circles, and in the diagram of the image surface the corresponding lines are also shown full and dotted respectively. The points of intersection are indicated by the series of numbers: 1, 2, . . . 20. In the diagram of the image magnified 200-fold, the corresponding points of intersection for the yellow D rays are numbered 1, 2, . . . 20, and those for the blue F rays 1′, 2′, . . . 20′.

In fig. 58, Plate II, the form of the diffused image at a distance of 308·648 mm. from the last centre plane is shown magnified 200 times. The co-ordinates of the points of intersection of the rays with this plane are indicated in the small table (XVI) in precisely the same manner as in the preceding table.

In this diagram the intersection points of the individual rays are indicated in the same manner as in fig. 56, Plate II. It will be easily seen that the diffused image in this plane is considerably smaller than in the first one, and that the crossing of the rays has not yet occurred.

APPENDIX

TABLE OF OBJECT AND IMAGE DISTANCES CORRESPONDING WITH VARIOUS MAGNIFICATIONS WHEN THE FOCAL LENGTH IS UNITY

I Minification.	II Image Distance.	III Object Distance.	I Minification.	II Image Distance.	III Object Distance.
1 fold	2·00	2·00	6·5	1·15	7·50
1·1	1·91	2·10	7·0	1·14	8·00
1·2	1·83	2·20	7·5	1·13	8·50
1·3	1·77	2·30	8·0	1·12	9·00
1·4	1·72	2·40	8·5	1·12	9·50
1·5	1·67	2·50	9·0	1·11	10·00
1·6	1·62	2·60	9·5	1·10	10·50
1·7	1·59	2·70	10·0	1·10	11·00
1·8	1·56	2·80	11·0	1·09	12·00
1·9	1·53	2·90	12·0	1·08	13·00
2·0	1·50	3·00	13·0	1·08	14·00
2·1	1·48	3·10	14·0	1·07	15·00
2·2	1·45	3·20	15·0	1·07	16·00
2·3	1·43	3·30	16·0	1·06	17·00
2·4	1·42	3·40	18·0	1·06	19·00
2·5	1·40	3·50	20·0	1·05	21·00
2·6	1·38	3·60	22·0	1·04	23·00
2·7	1·37	3·70	24·0	1·04	25·00
2·8	1·36	3·80	26·0	1·04	27·00
2·9	1·34	3·90	28·0	1·04	29·00
3·0	1·33	4·00	30·0	1·03	31·00
3·2	1·31	4·20	35·0	1·03	36·00
3·4	1·29	4·40	40·0	1·02	41·00
3·6	1·28	4·60	45·0	1·02	46·00
3·8	1·26	4·80	50·0	1·02	51·00
4·0	1·25	5·00	60·0	1·02	61·00
4·5	1·22	5·50	70·0	1·01	71·00
5·0	1·20	6·00	80·0	1·01	81·00
5·5	1·18	6·50	90·0	1·01	91·00
6·0	1·17	7·00	100·0	1·01	101·00
Magnification. I	Object Distance. II	Image Distance. III	Magnification. I	Object Distance. II	Image Distance. III

Use of the Table

The *largest* cross dimension of the object, and the *largest* cross dimension of the corresponding image should be expressed in the same unit. If the image is larger than the object, the case is one of magnification, and reference must then be made to the *lower* ends of the columns. If the image is smaller than the object, the case is one of minification, and reference must be made to the *tops* of the columns. If the object and image are equally large, the object appears of natural size, that is, the magnification and minification are both unity.

To determine what is the magnification or minification, the larger number should be divided by the smaller. The position of this number should then be noted in Column 1, and the corresponding image and object distances be read in Columns 2 and 3, opposite the corresponding number in Column 1. If these distances are then *multiplied by the focal length of the objective*, the required actual distances to the object and image will be obtained.

For example, suppose it is desired to reduce a card from a diameter of 40 cm. to a diameter of 25 cm. Since the image is smaller than the object, the case is one of minification, the actual figure being 1·6 times. Opposite the number 1·6 in Column 1 there will be found in Column 2 the image distance 1·62, and in Column 3 the object distance 2·60. If the focal length is 40 cm., then the image distance will be 1·62 × 40 = 64·8 cm., and the object distance 2·6 × 40 = 104 cm.

Or, for example, suppose it is desired to focus upon an object at a distance of 6·4 m. when the focal length of the objective is 40 cm. The object distance will then be $\frac{640}{40} = 16$ times the focal length. Opposite the number 16 in Column 3 there will be found in Column 2 the image distance, namely 1·07 times the focal length, that is 42·8 cm., and in Column 1 there will be found a minification of the image of 15 times.

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